

Approximate Reasoning in Fuzzy Systems Based on Pseudo-analysis and Uninorm Residuum

Márta Takács

Department of Information Technology, Budapest Tech
Népszínház utca 8, H-1081 Budapest, Hungary
takacs.marta@nik.bmf.hu

Abstract: The paper introduces novel residuum-based reasoning systems in a pseudo—analysis based uninorm environment. Based on the definitions and theorems for lattice ordered monoids and left continuous uninorms and t-norms, certain distance-based operators are focused on, with the help of which the uninorm-residuum based approximate reasoning system becomes possible in Fuzzy Logic Control (FLC) systems, but as it will be shown, this type of the reasoning partially satisfies the conditions for approximate reasoning and inference mechanism for FLC systems.

Keywords: FLC, approximate reasoning, uninorms

1 Introduction

The modelling of the uncertain processes in our modern society is a very complex problem. Since the systems are multi-criterial and multipart, decision processes become increasingly vague and hard to analyse. The human brain possesses some special characteristics that enable it to learn and reason in a vague and fuzzy environment. Naturally, there have been models that have investigated the behaviour of complex systems before the introduction of the fuzzy systems, but fuzzy systems have proven to be to a greater or lesser degree more useful than the classical models.

The real time functioning of the dynamic engineering system is a necessary condition. The earlier, differential-equation based models, with the expansion of the complexity of the real systems are growing out of all proportion, and it is very hard to construct a real time decision making model under those circumstances. Since the fuzzy logic control systems are based on linguistic variables [14], and fuzzy approximate reasoning, and therefore they are successful and effective [15].

Generally, the fundamental of the decision making in fuzzy based real systems is the approximate reasoning, which is a rule-based system. Knowledge representation in a rule-based system is done by means of IF...THEN rules. Furthermore, approximate reasoning systems allow fuzzy inputs, fuzzy antecedents, fuzzy consequents. “Informally, by approximate or, equivalently, fuzzy reasoning, we mean the process or processes by which a possibly imprecise conclusion is deduced from a collection of imprecise premises. Such reasoning is, for the most part, qualitative rather than quantitative in nature and almost all of it falls outside of the domain of applicability of classical logic”, [13]. This fuzzy representation allows a closer match with many of the important concepts of practical affairs, which lack the sharp boundaries assumed by classical logic.

The fundamental point arising from the classical two-valued logic is that it imposes a dichotomy on any mathematical model. However, in many cases of daily life, a borderline between the two possibilities is not evident. There are, of course the generalizations of two-valued logic, the multi-valued logics. In this model there may be a finite or infinite number of truth values, that is, an infinite number of degrees to which a property may be present. But in contrast with multi-valued logics, “fuzzy logic differs from conventional logical systems in that it aims at providing a model for approximate rather than precise reasoning”, [1].

Fuzzy logic also provides a system that is sufficiently flexible and expressive to serve as a natural framework for the semantics of natural languages. Also in fuzzy logic truth itself is allowed to be such as “quite true”, “more or less true”, [2]. Although application-oriented fuzzy systems seek to be simple and comprehensible, it is obvious that they are heavily related to the fields of classical multi-valued logic, operation research and functional analysis. There are numerous models that are yet to gain exact mathematical description, but have already proven their applicability in practice.

This latter question seems to be answered. Experts are confident in using special types of operations in fuzzy systems, such as t-norms, t-conorms, uninorms, and more generally, aggregation operators, and researchers are more and more meticulous in providing exact mathematical definitions for those. The focus is on certain properties of operators (continuity and representability, for instance), and those classes of operators are highlighted that correspond to the applications. This paper is a step towards the investigation of this problem by reviewing a specific case, where the investigated structure is a real semi-ring with pseudo-operations [16].

The question raised previously was whether there are general operation groups which satisfy the residuum-based approximate reasoning, but at the same time are easily comprehensible and acceptable to application-oriented experts. How come the residuum-based approximate reasoning is not as wide-spread in mathematical logic as the Mamdani-type? Why is the application of operations in fuzzy systems limited to that of the minimum and the product? In [11] only a partial answer to

this question is laid out. The axiom system presented in [11] greatly contributes to this. Situated between the theses, these axioms declare the expectations of the approximate reasoning systems, and it becomes clear to what degree they satisfy or violate this system.

The theoretical basis used in [11] provided by [7], which is currently the leading work in the world concerning fuzzy operators. It represents the monography describing t-norms, t-conorms and related operators for fuzzy sets and numbers. The other basic background of this research are the distance-based operators introduced by Rudas in his work [8],[9]. The characteristics of those operators were further investigated by Rudas and the author of this paper in various joint projects [10]. The research heavily relies on approximate reasoning and fuzzy logic theory (focusing on implications). The following works were mainly used in this investigation: [2], [4].

Concerning the structure of the work is the next: the first section an overview of pseudo-analysis based uninorm operators is given, with its role in residuum-based approximate reasoning. It is emphasized which operators from distance based operator group are the ones corresponding to the uninorm-based residuum known so far. The next section also introduces novel reasoning systems in a uninorm environment. Based on the theorems shown in the first section, certain distance-based operators are focused on, with the help of which the residuum-based approximate reasoning system becomes possible.

2 Mathematical Background of the Novel Approximate Reasoning Method

2.1 Real Semiring and the Pseudo-operators

The base for the pseudo-analysis is a real semiring, defined in the following way:

Let $[a, b]$ be a closed subinterval of $[-\infty, +\infty]$ (in some cases semi-closed subintervals will be considered) and let $\preceq$ be a total order on $[a, b]$. A *semiring* is the structure $(\preceq, \oplus, \otimes)$ if the following hold:

- $\oplus$ is pseudo-addition, i.e., a function $\oplus : [a, b] \times [a, b] \rightarrow [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element denoted by $\mathbf{0}$;
- $\otimes$ is pseudo-multiplication, i.e., a function $\otimes : [a, b] \times [a, b] \rightarrow [a, b]$ which is commutative, positively non-decreasing ($x \preceq y$ implies

$x \otimes z \leq x \otimes y$ where $z \in [a, b]_+ = \{z \mid z \in [a, b], \mathbf{0} \leq z\}$ associative and for which there exists a unit element denoted by $\mathbf{1}$.

- $\mathbf{0} \otimes z = \mathbf{0}$
- $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

Three basic classes of semirings with continuous (up to some points) pseudo-operations are:

- (i) The pseudo-addition is an idempotent operation and the pseudo-multiplication is not.
- (ii) Semi-rings with strict pseudo-operations defined by a monotone and continuous generator function $g : [a, b] \rightarrow [0, +\infty]$, i.e., g -semirings $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \otimes y = g^{-1}(g(x)g(y))$.
- (iii) Both operations, $\oplus$ and $\otimes$, are idempotent.

More on this structure can be found in [16], [17].

In this paper we will consider the interval $[a, b]$ to be the unit interval $[0, 1]$, in which case pseudo-operations will be t-conorm, t-norm and uninorm.

2.2 Uninorms

Both the neutral element 1 of a t-norm and the neutral element 0 of a t-conorm are boundary points of the unit interval. However, there are many important operations whose neutral element is an interior point of the underlying set. The fact that the first three axioms (commutativity, associativity, monotonicity) coincide for t-norms and for t-conorms, i.e., the only axiomatic difference lies in the location of the neutral element, has led to the introduction of a new class of binary operations closely related to t-norms and t-conorms.

A *uninorm* is a binary operation U on the unit interval, i.e., a function $U : [0, 1]^2 \rightarrow [0, 1]$ which satisfies the following properties for all $x, y, z \in [0, 1]$

- $U(x, y) = U(y, x)$, i.e. the uninorm is commutative,
- $U(U(x, y), z) = U(x, U(y, z))$, i.e. the uninorm is associative,
- $x \leq y \Rightarrow U(x, z) \leq U(y, z)$, i.e. the uninorm monotone,
- $U(e, x) = x$, i.e., a neutral element exists, which is $e \in [0, 1]$.

2.2.1 Lattice Ordered Monoids and Left Continuous Uninorms and T-norms

Let L be a non-empty set. Lattice is a partially (totally) ordered set which for any two elements $x, y \in L$ also contains their *join* $x \vee y$ (i.e., the least upper bound of the set $\{x, y\}$), and their *meet* $x \wedge y$ (i.e., the greatest lower bound of the set $\{x, y\}$), denoted by $(L, \preceq)$. Secondly, $(L, *)$ is a semi-group with the neutral element. Following [13], [14] let the following be introduced:

Definition 2.1.

Let $(L, \preceq)$ be a lattice and $(L, *)$ a semi-group with the neutral element.

- (i) The triple $(L, *, \preceq)$ is called a *lattice-ordered monoid* (or an *l-monoid*) if for all $x, y, z \in L$ we have

$$(LMI) \quad x * (y \vee z) = (x * y) \vee (x * z) \quad \text{and}$$

$$(LM2) \quad (x \vee y) * z = (x * z) \vee (y * z).$$

- (ii) An $(L, *, \preceq)$ *l-monoid* is said to be *commutative* if the semi-group $(L, *)$ is commutative.

- (iii) A commutative $(L, *, \preceq)$ *l-monoid* is said to be *commutative, residuated l-monoid* if there exists a further binary operation $\rightarrow_*$ on L , i.e., a function $\rightarrow_*: L^2 \rightarrow L$ (*the *residuum*), such that for all $x, y, z \in L$ we have

$$(Res) \quad x * y \preceq z \text{ if and only if } x \preceq (y \rightarrow_* z).$$

- (iv) An *l-monoid* $(L, *, \preceq)$ is called an *integral* if there is a greatest element in the lattice $(L, \preceq)$ (often called the universal upper bound) which coincides with the neutral element of the semi-group $(L, *)$.

Obviously, each *l-monoid* $(L, *, \preceq)$ is a partially ordered semi-group, and in the case of commutativity the axioms (LMI) and (LM2) are equivalent.

In the following investigations the focus will be on the lattice $([0, 1], \preceq)$, we will usually work with a complete lattice, i.e., for each subset A of L its join $\bigvee A$ and its $\bigwedge A$ exist and are contained in L . In this case, L always has a greatest element, also called the *universal upper bound*.

Example 2.1. If we define $*$: $[0, 1]^2 \rightarrow [0, 1]$ by

$$x * y = \begin{cases} \min(x, y) & \text{if } x + y \leq 1 \\ \max(x, y) & \text{otherwise} \end{cases}$$

then $([0,1], *, \leq)$ is a commutative, residuated l -monoid, and the $*$ -residuum is given by

$$x \rightarrow_* y = \begin{cases} \max(1-x, y) & \text{if } x \leq y \\ \min(1-x, y) & \text{otherwise} \end{cases}.$$

It is not an integral, since the neutral element is 0.5.

The operation $*$ results in an *uninorm*, and special types of distance based operators (see [9], [2] and section 4. from the paper).

The following result is an important characterization of left-continuous uninorms.

Theorem 2.1.

For each function $U : [0,1]^2 \rightarrow [0,1]$ the following are equivalent:

- (i) $([0,1], U, \leq)$ is a commutative, residuated l -monoid, with a neutral element
- (ii) U is a left continuous uninorm.

In this case the U -residuum $\rightarrow_U$ is given by

$$(ResU) \quad x \rightarrow_U y = \sup\{z \in [0,1] \mid U(x, z) \leq y\}.$$

Proof. (In the [9])

The work of De Baets, B. and Fodor, J. [2] presents general theoretical results related to residual implicators of uninorms, based on residual implicators of t-norms and t-conorms.

Residual operator R_U , considering the uninorm U , can be represented in the following form:

$$R_U(x, y) = \sup\{z \in [0,1] \mid U(x, z) \leq y\}.$$

Uninorms with the neutral elements $e = 0$ and $e = 1$ are t-norms and t-conorms, respectively, and related residual operators are widely discussed, we also find suitable definitions for uninorms with neutral elements $e \in]0,1[$.

If we consider a uninorm U with the neutral element $e \in]0,1[$, then the binary operator R_U is an impicator if and only if $(\forall z \in]e,1[)(U(0, z) = 0)$. Furthermore R_U is an impicator if U is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying $(\forall z \in [0,1])(g(z) = 0 \Leftrightarrow z = 1)$.

The residual impicator R_U of uninorm U can be denoted by Imp_U .

Consider a uninorm U , then R_U is an implicator in the following cases:

- (i) U is a conjunctive uninorm,
- (ii) U is a disjunctive representable uninorm,
- (iii) U is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying

$$(\forall z \in [0,1])(g(z) = 0 \Leftrightarrow z = 1).$$

3 Approximate Reasoning and the Fuzzy Logic Control

In control theory and also in theory of the approximate reasoning introduced by Zadeh in 1979, [11] much of the knowledge of system behaviour and system control can be stated in the form of if-then rules. The Fuzzy Logic Control, FLC has been carried out searching for different mathematical models in order to supply these rules.

In most sources it was suggested to represent an

if x is A then x is B

rule in the form of fuzzy implication (shortly $Imp(A,B)$, relation (shortly $R(A,B)$), or simply as a connection (for example as a t-norm, $T(A,B)$) between the so called rule premise: x is A and rule consequence: y is B . Let x be from universe X , y from universe Y , and let x and y be linguistic variables. Fuzzy set A in X is characterised by its membership function $\mu_A: x \rightarrow [0,1]$. The most significant differences between the models of FLC-s lie in the definition of this connection, relation or implication.

The other important part of the FLC is the inference mechanism. One of the widely used methods is the Generalised Modus Ponens (GMP), in which the main point is, that the inference y is B' is obtained when the propositions are:

- the i^{th} rule from the rule system of n rules: if x is A_i then y is B_i
- and the system input x is A' .

GMP sees the real influences of the implication or connection choice on the inference mechanisms in fuzzy systems ([3],[10]). Usually the general rule consequence for i^{th} rule from a rule system is obtained by

$$B'_i(y) = \sup_{x \in X} T(A'(x), Imp(A_i(x), B_i(y))).$$

3.1 The Axioms of Inference Mechanism

Let RB a fuzzy rule system of n rules, with rule premises x is A_i and rule consequences: y is B_i ($i = 1, 2, \dots, n$). Let x be from universe X , y from universe Y , and let x and y be linguistic variables. Fuzzy set A_i on $X \subset \mathfrak{R}$ finite universe is characterized by its membership function $\mu_{A_i}: x \rightarrow [0, 1]$, and fuzzy set B on $Y \subset \mathfrak{R}$ universe is characterized by its membership function $\mu_{B_i}: y \rightarrow [0, 1]$. Let x is A' be the system input, where A' is characterized by its membership function $\mu_{A'}: x \rightarrow [0, 1]$.

Applying the generalized compositional rule of inference to given components, the i -th rule output, with respect to the given RB and given system input A' , is y is B'_i given by the expression

$$B'_i(y) = \sup_{x \in X} T(A'(x), \text{Imp}(A_i(x), B_i(y))),$$

where, on a general level, Imp is the relationship between rule base premise and rule base consequence, satisfying the following conditions:

(out1) If the input coincides with one of the premises, then the resulting output coincides with the corresponding consequence, i.e.,

$$(\exists i \in \{1, 2, \dots, n\})(A' = A_i) \text{ then } B'_i = B_i.$$

(out2) For each normal input A' the output is not contained in all consequences, i.e.,

$$(\exists i \in \{1, 2, \dots, n\})(B'_i < B_i).$$

(out3) The rule output belongs to the convex hull of B_i , ($i \in I$), where

$$I = \{i | 1 \leq i \leq n, \text{Supp}(A') \cap \text{Supp}(A_i) \neq \emptyset\}.$$

In [9] we can find an axiom system on the same principle.

4 Residuum-based Approximate Reasoning with Distance-based Uninorms

In fact the uninorms offer new possibilities in fuzzy approximate reasoning, because the low level of covering over of rule premise and rule input has measurable influence on rule output as well. In some applications the meaning of that novel t-norms, has practical importance. The modified Mamdani's approach, with similarity measures between rule premises and rule input, does not rely on

the compositional rule inference any more, but still satisfies the basic conditions supposed for the approximate reasoning for a fuzzy rule base system [15].

Having results from [2], we can introduce residuum-based inference mechanism using distance-based uninorms.

4.1 Modified Distance-based Operators

The distance-based operators can be expressed by means of the min and max operators as follows (the only modification on distance based operators described in [9] is the boundary condition for neutral element e):

the *maximum distance minimum operator* with respect to $e \in [0,1]$ is defined as

$$\max_e^{\min} = \begin{cases} \max(x, y), & \text{if } y > 2e - x \\ \min(x, y), & \text{if } y < 2e - x, \\ \min(x, y), & \text{if } y = 2e - x \end{cases}$$

the *minimum distance minimum operator* with respect to $e \in [0,1]$ is defined as

$$\min_e^{\min} = \begin{cases} \min(x, y), & \text{if } y > 2e - x \\ \max(x, y), & \text{if } y < 2e - x. \\ \min(x, y), & \text{if } y = 2e - x \end{cases}$$

The distance-based operators have the following properties

$\max_e^{\min}$ and $\max_e^{\max}$ are uninorms,

the dual operator of the uninorm $\max_e^{\min}$ is $\max_{1-e}^{\max}$, and

the dual operator of the uninorm $\max_e^{\max}$ is $\max_{1-e}^{\min}$.

Based on results from [2] and [3] we conclude:

Operator $\max_{0.5}^{\min}$ is a conjunctive left-continuous idempotent uninorm with neutral element $e \in [0,1]$ with the super-involutive decreasing unary operator $g(x) = 2e - x = 2 \cdot 0.5 - x \Rightarrow g(x) = 1 - x$.

Operator $\min_{0.5}^{max}$ is a disjunctive right-continuous idempotent uninorm with neutral element $e \in]0,1]$ with the sub-involutive decreasing unary operator $g(x) = 2e - x = 2 \cdot 0.5 - x \Rightarrow g(x) = 1 - x$.

4.2 Idempotent Uninorms and the Residual Implicators of Uninorms

A binary operator V is called idempotent, if $V(x, x) = x, (\forall x \in X)$. It is well known, that the only idempotent t-norm is $\min$, and the only t-conorm is $\max$.

In [2], by De Baets, B. and Fodor, J., has studied two important classes of uninorms: the class of left-continuous and the class of right-continuous ones.

If we suppose a unary operator g on set $[0,1]$, then g is called super-involutive if $g(g(x)) \geq x$ for $(\forall x \in [0,1])$.

A binary operator U is a conjunctive left-continuous idempotent uninorm with neutral element $e \in]0,1]$ if and only if there exist a super-involutive decreasing unary operator g with fixpoint e and $g(0) = 1$ such that U for any $\forall (x, y) \in [0,1]^2$ is given by

$$U(x, y) = \begin{cases} \min(x, y) & \text{if } y \leq g(x) \\ \max(x, y) & \text{elsewhere} \end{cases}.$$

Residual operator R_U , considering the uninorm U , can be represented in the following form:

$$R_U(x, y) = \sup\{z \in [0,1] \mid U(x, z) \leq y\}.$$

Uninorms with neutral elements $e = 0$ and $e = 1$ are t-norms and t-conorms, respectively, and related residual operators are widely discussed.

If we consider a uninorm U with neutral element $e \in]0,1[$, then the binary operator R_U is an implicator if and only if $(\forall z \in]e,1[)(U(0, z) = 0)$. Furthermore R_U is an implicator if U is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying $(\forall z \in [0,1])(g(z) = 0 \Leftrightarrow z = 1)$.

The residual implicator R_U of uninorm U can be denoted by Imp_U .

4.3 Residual Implicators of Distance Based Operators

According to Theorem 8. in [2] we introduce impicator of distance based operator $\max_{0.5}^{\min}$.

Consider the conjunctive left-continuous idempotent uninorm $\max_{0.5}^{\min}$ with the unary operator $g(x) = 1 - x$, then its residual impicator $\text{Imp}_{\max_{0.5}^{\min}}$ is given by

$$\text{Imp}_{\max_{0.5}^{\min}} = \begin{cases} \max(1-x, y) & \text{if } x \leq y \\ \min(1-x, y) & \text{elsewhere} \end{cases} \quad (1)$$

4.4 Residuum-based Approximate Reasoning with Distance Based Operator

Although the minimum plays an exceptional role in fuzzy control theory, there are situations requiring new models. In system control one would intuitively expect to make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. The distance-based operators group satisfy these properties. Let us consider a mathematical approach: residuum-based approximate reasoning and inference mechanism. Hence, and because of the results from sections of this paper we can consider the general rule consequence for i -th rule from a rule system as

$$B_i'(y) = \sup_{x \in X} \left(\max_{0.5}^{\min} \left(A'(x), \text{Imp}_{\max_{0.5}^{\min}}(A_i(x), B_i(y)) \right) \right)$$

or, using formula (3.1.)

$$B_i'(y) = \sup_{x \in X} \begin{cases} \max_{0.5}^{\min}(A'(x), \max(1-A_i(x), B_i(y))) & \text{if } A_i(x) \leq B_i(y) \\ \max_{0.5}^{\min}(A'(x), \min(1-A_i(x), B_i(y))) & \text{elsewhere} \end{cases} \quad (2)$$

The rule base output is constructed as a crisp value calculated with a defuzzification model, from rule base output. Rule base output is an aggregation of all rule consequences $B_i'(y)$, from the rule base. As aggregation operator, in this case, dual operator $\max_{0.5}^{\max}$ of $\max_{0.5}^{\min}$ can be used.

$$B'_{out}(y) = \max_{0.5}^{\max}(B_n'(y), \max_{0.5}^{\max}(B_{n-1}'(y), \max_{0.5}^{\max}(\dots, \max_{0.5}^{\max}(B_2'(y), B_1'(y)))).$$

4.5 Uninorm-residuum Based Approximate Reasoning and the Axioms of Inference Mechanism

Axiom (out1): Taken into account Proposition 13. from [2], it can be conclude, that conjunctive left-continuous idempotent uninorm $\max_{0.5}^{min}$ and its implicator $\text{Imp}_{\max_{0.5}^{min}}$ satisfy the inequality

$$B'_i(y) = \max_{0.5}^{min} \left(A'(x), \text{Imp}_{\max_{0.5}^{min}}(A_i(x), B_i(y)) \right) \leq B_i(y)$$

for i -th rule in rule base system, if $A'(x) = A_i(x)$ for all $x \in X$. It means, that this

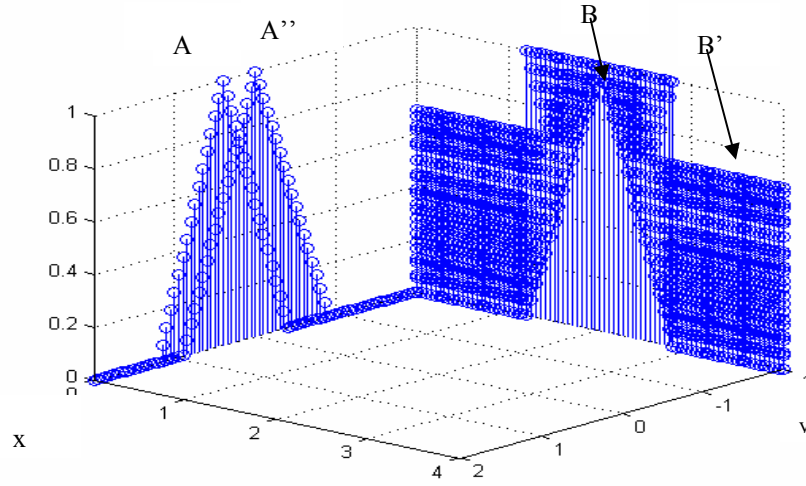


Figure 1

Output B' calculating with uninorm-residuum based approximate reasoning

type of reasoning partially satisfies the conditions for approximate reasoning, hence $B'_i(y) = B_i(y)$ or $B'_i(y) < B_i(y)$ if $A'(x) = A_i(x)$ for all $x \in X$.

Axiom (out2). In most of cases uninorm-residuum based approximate reasoning violates this axioms of inference mechanism, because for normal input A' the output is contained in all consequences, if we have not “faired” rule.

Axiom (out3): If $A \neq A'$, the rule output belongs not to the convex hull of B_i , ($i=1, n$).

Figure 1. shows the situation, where $A \neq A'$, calculating rule output B' with the expression (2).

In [8] it was proved in simulations, that in this case if all the rules, where the rule doesn't has real influence on output, have been real time eliminated, the results are acceptable.

Conclusions

Based on the definitions and theorems for lattice ordered monoids and left continuous uninorms and t-norms, certain distance-based operators are focused on, with the help of which the uninorm-residuum based approximate reasoning system becomes possible in Fuzzy Logic Control (FLC) systems. This type of reasoning partially satisfies the conditions for approximate reasoning and inference mechanism for FLC systems.

References

- [1] Bellmann, R.E., Zadeh, L.A., (1977), *Local and fuzzy logic*, Modern Uses of Multiple-Valued Logic, edited by Dunn, J.M., Epstein, G., Reidel Publ., Dordrecht, The Netherlands, pp. 103-165.
- [2] De Baets, B., Fodor, J., (1999), *Residual operators of uninorms*, Soft Computing 3., (1999), pp. 89-100.
- [3] Fodor, J., Rubens, M., (1994), *Fuzzy Preference Modeling and Multi-criteria Decision Support*. Kluwer Academic Pub., 1994.
- [4] G. Hancke, Á. Szeghegyi: „Nonlinear Control Via TP Model Transformation: The TORA System Example”, In the Proc.Of the Symposium on Applied Machine Intelligence (SAMI 2004), Herlany, Slovakia, Januar 16-17 2004, pp.333 - 340.
- [5] Klement, E.P., Mesiar, R, Pap, E., *'Triangular Norms'*, Kluwer Academic Publishers, 2000, ISBN 0-7923-6416-3
- [6] Rudas, I., *Absorbing-Norms*, in Proceedings of the 3th International Symposium of Hungarian Researchers Computational Intelligence, Budapest, Nov. 2002., pp. 25-43.
- [7] Rudas, Imre; Kaynak, O., New Types of Generalized Operations, Computational Intelligence, Soft Computing and Fuzzy-Neuro Integration with Applications, Springer NATO ASI Series. Series F, Computer and Systems Sciences, Vol. 192. 1998. (O. Kaynak, L. A. Zadeh, B. Turksen, I. J. Rudas editors), pp. 128-156.
- [8] M. Takacs: Investigation on a Special Group of Fuzzy Implication Operators and Fuzzy Inference Mechanism Using a Simplified Rule Base

- System, Proceedings of the 1997 International Conference on Neural Information Processing and Intelligent Information Systems, New Zealand, Dunedin, Univ. of Otago, 23-28 November 1997, Vol. 2, pp. 793-796.
- [9] Takacs, M., Approximate Reasoning in Fuzzy Systems Based On Pseudo-Analysis, Phd Thesis, Univ. of Novi Sad, 2004.
- [10] Zadeh, L. A., A Theory of approximate reasoning, In Hayes, J., and editors, Mashine Intelligence, Vol. 9, Halstead Press, New York, 1979., pp. 149-194.
- [11] Zadeh, L. A., (1999), From Computing with Numbers to Computing with Words – From Manipulation of Measurements to manipulation of Perceptions, In Proc. Of EUROFUSE –SIC Conf. 1999., Budapest, pp. 1-3.
- [12] Zimmermann, H.J., (1991), Fuzzy Sets, Decision Making and Expert Systems. Kluwer, Boston, 1991.
- [13] Pap, E., Triangular norms in modelling uncertainly, non-linearity and decision, in Proceedings of the 2th International Symposium of Hungarian researchers Computational Intelligence, ISBN 963 7154 06 X, pp. 7-18.
- [14] Birkhoff, G., (1973), Lattice theory. American Mathematical Society, Providence
- [15] Takacs, M., (2003c), Approximate reasoning with Distance-based Operators and degrees of coincidence, in Principles of Fuzzy Preference Modelling and Decision Making, edited by Bernard de Baets, János Fodor, Academia Press Gent, 2003., ISBN 90-382-0567-8
- [16] Pap, E., Ralević, N., (1999), Pseudo-operations on finite intervals, Novi Sad J. Math. Vol. 29 (1) (1999), pp. 9-18.
- [17] Pap, E., Štajner, I., (1999), Generalized pseudo-convolution in the theory of probabilistic metric spaces, information, fuzzy numbers, optimization, system theory, Fuzzy Sets and Systems 102., pp. 393-415.