

Level Crossing Probabilities of the Ornstein – Uhlenbeck Process

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Abstract: The Ornstein Uhlenbeck process is a Gaussian process X_t with independent increments and autocorrelation $E(X_t X_{t+s}) = \frac{e^{-|s|}}{2}$. First the Laplace transform of the probability density $P(X_t = x | X_0 = p)$ is computed. Using this, the Laplace transform of X_t first time reaching a given value x is derived. It is proved that these results agree with the special case derived earlier by Bellman and Harris (Pacific J. Math. 1, 1951).

1 Definitions

The Ornstein Uhlenbeck process is a stationary Gaussian-Markov process X_t such that the joint distribution of $X_{t_1}, X_{t_2} \dots X_{t_m}$ is a gaussian and is dependent only on the differences $t_j - t_i$ where $i < j$ and the autocorrelation function is given by

$$E(X_s \cdot X_{s+t}) = \frac{1}{2} e^{-|t|} \quad (1.1)$$

$$EX_t = 0 \text{ and } EX_t^2 = \frac{1}{2}. \quad (1.2)$$

Let X be a random vector with normal distribution, then the density of its probability distribution is:

$$\frac{1}{2\pi|\Sigma|} e^{-\frac{1}{2}X^T \Sigma^{-1} X}$$

where $X = \begin{pmatrix} X \\ Y \end{pmatrix}$ and Σ is the correlation matrix:

$$\begin{pmatrix} \rho_1 & \sigma \\ \sigma & \rho_2 \end{pmatrix}$$

with $\rho_1 = EX^2, \rho_2 = EY^2, \sigma_1 = EXY$ and $|\Sigma| = \rho_1\rho_2 - \sigma^2$. Clearly

$$\Sigma^{-1} = \frac{\begin{pmatrix} \rho_2 & -\sigma \\ -\sigma & \rho_1 \end{pmatrix}}{\rho_1\rho_2 - \sigma^2}.$$

Hence the joint probability density

$$P(X = x, Y = y) = \frac{1}{2\pi\sqrt{\rho_1\rho_2 - \sigma^2}} \exp\left(-\frac{\rho_2 x^2 - 2\sigma xy + \rho_1 y^2}{2(\rho_1\rho_2 - \sigma^2)}\right).$$

It follows from here that

$$\begin{aligned} P(Y = y | X = x) &= \frac{\frac{1}{2\pi\sqrt{\rho_1\rho_2 - \sigma^2}} \exp\left(-\frac{\rho_2 x^2 - 2\sigma xy + \rho_1 y^2}{2(\rho_1\rho_2 - \sigma^2)}\right)}{\frac{e^{-\frac{x^2}{2\rho_1}}}{\sqrt{2\pi\rho_1}}} \\ &= \frac{1}{\sqrt{2\pi\frac{\rho_1\rho_2 - \sigma^2}{\rho_1}}} \exp\left(-\frac{\left(y - \frac{\sigma^2}{\rho_1}x\right)^2}{2\frac{\rho_1\rho_2 - \sigma^2}{\rho_1}}\right). \end{aligned}$$

Applying this to what concerns us, the Ornstein-Uhlenbeck process, we can determine the probability density $P(X_t = x | X_0 = p)$.

Clearly

$$\rho_1 = \rho_2 = \frac{1}{2}, \sigma = \frac{e^{-t}}{2} \text{ so } \frac{2(\rho_1 \rho_2 - \sigma^2)}{\rho_1} = \frac{2\left(\frac{1}{2} \frac{1}{2} - \frac{e^{-2t}}{4}\right)}{\frac{1}{2}} = 1 - e^{-2t} \cdot \frac{\sigma}{\rho_1} = e^{-t}$$

Hence:

$$P(X_t = x | X_0 = p) = \frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}. \quad (1.3)$$

We shall denote this with $P(t, p, x)$ or $P(p, x)$ and call it the fundamental function. The special cases $p = 0$ and $x = 0$ are important also:

$$P(X_t = x | X_0 = 0) = \frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}. \quad (1.4)$$

$$P(X_t = 0 | X_0 = p) = \frac{e^{-\frac{p^2 e^{-2t}}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}. \quad (1.5)$$

By simple substitution it is easy to prove that (1.3) satisfies the forward equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial t} + u$$

and the backward equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial p^2} - p \frac{\partial u}{\partial p}.$$

This also implies that (1.4) satisfies the forward equation and (1.5) satisfies the backward equation.

2 The Laplace Transforms of the Fundamental Functions

Since both $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}$ and $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}$ satisfy the $u_t = \frac{1}{2}u_{xx} + u + xu_x$

forward equation their Laplace transforms must satisfy the $sU = \frac{U_{xx}}{2} + U + xU_x$ second order ordinary differential equation, that is the equation

$$U'' + 2xU' + 2(1-s)U = 0 \quad (2.1)$$

To find the solutions of (2.1) let us consider the confluent hypergeometric equation

$$xy'' = +(c-x)y' - ay = 0 \quad (2.2)$$

The two solutions of this are the:

$${}_1F_1(a, c; x) = 1 + \frac{a}{c} \frac{x}{1!} + \frac{a(a+1)x^2}{c(c+1)2!} \dots$$

and $x^{1-c} {}_1F_1(a+1-c, 2-c; x)$ Kummer functions. Let us consider the following transformation of (2.2) $u = y(kx^2)$ where k is an arbitrary nonzero constant.

Clearly:

$$\begin{aligned} u &= y(kx^2) \\ u' &= 2kxy' \\ u'' &= 2ky' + 4k^2x^2y''. \end{aligned}$$

Hence:

$$y = u$$

$$y' = \frac{u'}{2kx}$$

$$y'' = \frac{u'' - \frac{u'}{x}}{4k^2x^2}$$

Substituting these into (2.2) gives:

$$\frac{kx^2 \left(u'' - \frac{u'}{x} \right)}{4k^2x^2} + (c - kx^2) \frac{u'}{2kx} - au = 0$$

which in turn, after some simplification becomes:

$$u'' + \left(\frac{2c-1}{x} - 2kx \right) u' - 4kau = 0.$$

Putting $c = \frac{1}{2}$ gives: $u'' - 2kxu' - 4kau = 0$.

Let us compare this with (2.1)

$$U'' + 2xU' + 2(1-s)U = 0$$

$$-2k = 2$$

$$-4ka = 2(1-s).$$

Hence we get for k and for $ak = -1$ and $a = \frac{1-s}{2}$. Therefore the solutions of

$$(2.1) \text{ are } F_1 = F\left(\frac{1-s}{2}, \frac{1}{2}; -x^2\right) \text{ and } F_2 = xF\left(1 - \frac{s}{2}, \frac{3}{2}; -x^2\right).$$

Now we are in the position to determine the Laplace transform of $\frac{e^{-\frac{x^2}{1-e^{-2t}}}}{\sqrt{\pi}(1-e^{-2t})}$.

Clearly it must be of the form $AF_1 + xBF_2$ where A and B some constants. To this end Laplace transform will be evaluated for some special cases. The Laplace

transform of $\frac{e^{-\frac{x^2}{1-e^{-2t}}}}{\sqrt{\pi}(1-e^{-2t})}$ is clearly:

$$\int_0^\infty \frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} e^{-st} dt.$$

Writing t instead of e^{-t} transforms it into a Mellin type integral:

$$\int_0^1 \frac{e^{-\frac{x^2}{1-t^2}}}{\sqrt{\pi}(1-t^2)} t^{s-1} dt.$$

Substituting $\sqrt{t}$ instead of t yields

$$\frac{1}{2\sqrt{\pi}} \int_0^1 \frac{e^{-\frac{x^2}{1-t}}}{\sqrt{1-t}} t^{\frac{s}{2}-1} dt.$$

For $x = 0$ this becomes the beta function type integral:

$$\frac{1}{2\sqrt{\pi}} \int_0^1 \frac{t^{\frac{s}{2}-1}}{\sqrt{1-t}} dt = \frac{1}{2\sqrt{\pi}} B\left(\frac{1}{2}, \frac{s}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1+s}{2}\right)}.$$

Hence

$$A = \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)}.$$

Clearly A is the Laplace transform of $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$. To determine the value of

B let us consider the x derivative of the Laplace transform, which is:

$$-\int_0^1 \frac{x e^{-\frac{x^2}{1-t}}}{\sqrt{\pi}(1-t)^{\frac{3}{2}}} t^{\frac{s}{2}-1} dt.$$

In the present case we cannot take the $x \rightarrow 0$ limit by simply substituting

$x \rightarrow 0$ for x because $\frac{x e^{-\frac{x^2}{t}}}{\sqrt{\pi - t^{\frac{3}{2}}}}$ does not converge uniformly to 0 as $x \rightarrow 0$. In

fact it is a “delta function type function”, its integral being

$$\int_0^1 \frac{x e^{-\frac{x^2}{t}}}{\sqrt{\pi} t^{\frac{3}{2}}} dt = 1.$$

For it is known from theory of the heat equation that, for an arbitrary continuous function $f(t)$

$$\lim_{x \rightarrow 0} \int_0^t \frac{x}{\sqrt{\pi}} \frac{e^{-\frac{x^2}{t-r}}}{(t-r)^{\frac{3}{2}}} f(r) dr = \lim_{x \rightarrow 0} \int_0^t \frac{x}{\sqrt{\pi}} \frac{e^{-\frac{x^2}{r}}}{r^{\frac{3}{2}}} f(t-r) dr = f(t).$$

Hence in the present case:

$$-\lim_{x \rightarrow 0} \int_0^1 \frac{x e^{-\frac{x^2}{1-t}}}{\sqrt{\pi} (1-t)^{\frac{3}{2}}} t^{\frac{s}{2}-1} dt = t^{\frac{s}{2}-1} \Big|_{t=1} = -1,$$

thus $B = -1$. Therefore the Laplace transform of $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$ is

$$AF_1 - xF_2 = \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)} F\left(\frac{1-s}{2}, \frac{1}{2}; -x^2\right) - xF\left(1-\frac{s}{2}, \frac{3}{2}; -x^2\right).$$

Now we compute the The Laplace transform of $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$. It has been shown

that it satisfies the backward equation $u_t = -pu_p + \frac{u_{pp}}{2}$. Therefore its Laplace transform is the solution of the second order linear differential equation $sU = -pU_p + \frac{U_{pp}}{2}$ that is of the equation

$$U'' + 2pU' + 2sU = 0$$

Now the solution of $u'' - 2kxu' - 4kau = 0$ are $F\left(a, \frac{1}{2}; kx^2\right)$ and $x F\left(a + \frac{1}{2}, \frac{3}{2}; kx^2\right)$.

Comparing the two equations we get for k

$$2k = 2$$

$$4ka = 2s$$

that is $k=1$ and $a = \frac{s}{2}$. Thus the Laplace transform must be the linear combination of $G_1 = F\left(\frac{s}{2}, \frac{1}{2}; p^2\right)$ and $pG_2 = pF\left(\frac{1+s}{2}, \frac{3}{2}; p^2\right)$. To find the conficciens of G_1 and pG_2 let us inspect the Laplace transform itself.

$$\int_0^\infty \frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} e^{-st} dt.$$

Writing t instead of e^{-t} it transforms again into the Mellin type integral:

$$\int_0^1 \frac{e^{-\frac{p^2 t^2}{(1-t^2)}}}{\sqrt{\pi}(1-t^2)} t^{s-1} dt.$$

Substituing $\sqrt{t}$ instead of t yields

$$\frac{1}{2\sqrt{\pi}} \int_0^1 \frac{e^{-\frac{p^2 t^2}{1-t}}}{\sqrt{1-t}} t^{\frac{s}{2}-1} dt.$$

Again putting $p = 0$ this becomes:

$$\frac{1}{2\sqrt{\pi}} \int_0^1 \frac{t^{\frac{s}{2}}}{\sqrt{1-t}} dt = A.$$

The coefficient of pG_2 can be evaluated the same way as was done for

$$\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} \text{ and it is found to be again } -1. \text{ Thus the Laplace transform of } \frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} \text{ is}$$

$$AG_1 - pG_2 = AF\left(\frac{s}{2}, \frac{1}{2}; p^2\right) - pF\left(1 - \frac{1+s}{2}, \frac{3}{2}; p^2\right).$$

The above result can be arrived at directly from the Laplace transform of

$$\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}.$$

To this end let us inspect

$$\int_0^\infty \frac{e^{-\frac{p^2 e^{-2t}}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} e^{-st} dt$$

using

$$\frac{p^2 e^{-2t}}{1-e^{-2t}} = \frac{p^2}{1-e^{-2t}} - p^2.$$

This becomes $e^{p^2} \int_0^\infty \frac{e^{-\frac{p^2}{1-e^{-2t}}}}{\sqrt{\pi}(1-e^{-2t})} e^{-st} dt$ and the integra here is of the same form as of the Laplace transform of $\frac{e^{-\frac{x^2}{1-e^{-2t}}}}{\sqrt{\pi}(1-e^{-2t})}$ except we have p instead of x .

Therefore the Laplace transform of $\frac{e^{-\frac{x^2}{1-e^{-2t}}}}{\sqrt{\pi}(1-e^{-2t})}$ is

$$e^{p^2} \left(\frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)} F\left(\frac{1-s}{2}, \frac{1}{2}; -p^2\right) - p F\left(1 - \frac{s}{2}, \frac{3}{2}; -p^2\right) \right)$$

Applying Kummer's formula $F(a, c; x) = e^x F(c-a, c; x)$ we get for the Laplace

transform of $\frac{e^{-\frac{p^2 e^{-2t}}{1-e^{-2t}}}}{\sqrt{\pi}(1-e^{-2t})}$

$$\frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)} F\left(\frac{s}{2}, \frac{1}{2}; p^2\right) - p F\left(1 - \frac{1+s}{2}, \frac{3}{2}; p^2\right).$$

3 The Laplace Transforms of $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$

We have seen that the $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$ fundamental function satisfies both the forward and backward equations, therefore its Laplace transform must satisfy both of the ordinary differential equations:

$$U'' + 2pU' + 2(1-s)U = 0 \quad (3.1)$$

$$U'' - 2pU' - 2sU = 0. \quad (3.2)$$

Because of (3.1) must be of the form: $H F_1 + K x F_2$, where H and K must be some linear combinations of G_1 and $p G_2$ since it satisfies (3.2) as well. Let us

observe that $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$ is analytic in x for all values p and t except when $t = 0$ and $x = p$, in the latter case it is undefined. Therefore its Laplace transform

is analytic in the $x \leq p$ domain as well. Putting $x = 0$ in $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$ gives

$\frac{e^{-\frac{p^2 e^{-t^2}}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$ and we have seen that its Laplace transform is $A G_1 - p G_2$, so

$H = A G_1 - p G_2$ (when $x \leq p$). The determination of K is more involved. Differentiating the fundamental function by x gives:

$$\frac{2pe^{-t} e^{-\frac{p^2 e^{-t^2}}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})^{\frac{3}{2}}} = e^{p^2} \frac{2pe^{-t} e^{-\frac{p^2 e^{-t^2}}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})^{\frac{3}{2}}}. \quad (3.3)$$

Clearly the coefficient K is the Laplace transform of (3.3). To evaluate it let us compute the following convolution integral:

$$e^{p^2} \frac{2pe^{-t} e^{-\frac{p^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})^{\frac{3}{2}}} * \frac{1}{\sqrt{\pi}(1-e^{-2t})}. \quad (3.4)$$

It has been shown that the Laplace transform of the second factor in (3.4) is A , so the Laplace transform of (3.3) is the Laplace transform of (3.4) divided into A . Next we evaluate (3.4):

$$\int_0^t e^{p^2} \frac{2pe^{-r} e^{-\frac{p^2}{(1-e^{-2r})}}}{\sqrt{\pi}(1-e^{-2r})^{\frac{3}{2}}} \frac{1}{\sqrt{\pi}(1-e^{-2(t-r)})} dr =$$

putting r for e^{-r} yields:

$$\int_T^1 e^{p^2} \frac{2pre^{-\frac{p^2}{(1-r^2)}}}{\sqrt{\pi}(1-r^2)^{\frac{3}{2}}} \frac{1}{\sqrt{\pi}(r^2-T^2)} dr =$$

where $T = e^{-t}$. Substituting $\sqrt{r}$ for r gives:

$$\begin{aligned} & e^{p^2} \int_{T^2}^1 \frac{pe^{-\frac{p^2}{(1-r)}}}{\sqrt{\pi}(1-r)^{\frac{3}{2}}} \frac{1}{\sqrt{\pi}(r^2-T^2)} dr \\ &= e^{p^2} \int_0^{1-T^2} \frac{pe^{-\frac{p^2}{(1-T^2-r)}}}{\sqrt{\pi}(1-T^2-r)^{\frac{3}{2}}} \frac{1}{\sqrt{\pi r}} dr \\ &= e^{-p^2} \cdot \frac{pe^{\frac{p^2}{t}}}{\sqrt{\pi t^3}} * \frac{1}{\sqrt{\pi t}} \Big|_{t=1-T^2} = \frac{e^{-\frac{p^2 e^{-2t}}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}}. \end{aligned}$$

Thus we have for the coefficient $K = \frac{AG_1 - pG_2}{A}$. Hence the Laplace transform

$$\text{of } \frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} \text{ is for } x \leq p:$$

$$(AG_1 - pG_2)F_1 + xF_2 \frac{AG_1 - pG_2}{A} = \frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A}.$$

Next let us consider the case $p \leq x$. If the same computation is repeated but instead of $x = 0$ we look at $p = 0$, that is we compute the coefficients of G_1 and

pG_2 . Putting $p = 0$ in $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$ gives $\frac{e^{-\frac{x^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$. Its Laplace transform is $AF_1 - xF_2$, carrying through similar computation as was done for the coefficient of xF_2 we get for the coefficient for pG_2 $\frac{AF_1 - xF_2}{A}$. Thus the

Laplace transform of $\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}$ when $p \leq x$ is:

$$\frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A}. \text{ Hence the Laplace transform of } \frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} \text{ is:}$$

$$? \left(\frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} \right) = \begin{cases} \frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A} & \text{if } p \leq x \\ \frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A} & \text{if } x \leq p. \end{cases} \quad (3.5)$$

4 Level Crossing Probabilities

Let the random variable FC or $FC(x)$ (first crossing) be the smallest possible value of t such that $X_t = x$ given $X_0 = p$. Let $\varphi(t, p, x)$ be the distribution of FC, clearly: $\varphi(t, p, x) * P(t, x, x) = P(t, p, x)$.

$$\text{That is: } \varphi(t, p, x) * \frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})} = \frac{e^{-\frac{(x-pe^{-t})^2}{(1-e^{-2t})}}}{\sqrt{\pi}(1-e^{-2t})}.$$

Now the probability that X_t stays below x is: $P\left(\sup_{0 \leq r \leq t} X_r\right) = 1 - \int_0^t \varphi(r) dr$.

Let us denote the Laplace transform of φ by Ψ , then Ψ for $0 \leq p \leq x$ using (3.5) can be expressed as

$$\Psi = \frac{(AG_1(p) + pG_2(p))(AF_1(x) + xF_2(x))}{(AG_1(x) + xG_2(x))(AF_1(x) + xF_2(x))} \quad (4.1)$$

$$= \frac{AG_1 + pG_2}{AG_1 + xG_2} \quad (4.2)$$

For the special case when $p = 0$, that is when X_t reaches level x subject to the initial condition $X_0 = 0$ is

$$\Psi = \frac{A}{AG_1 + xG_2}. \quad (4.3)$$

For this case Bellman and Harris [1] found the following expression:

$$\Psi = \frac{\frac{1}{2} \Gamma\left(\frac{s}{2}\right)}{\int_0^\infty e^{-y^2 + 2xy} y^{s-1} dy}. \quad (4.4)$$

For the case $p \geq x \geq 0$:

$$\Psi(p, x) = \frac{\frac{(AF_1 + xF_2)(AG_1 - pG_2)}{A}}{\frac{(AF_1 + xF_2)(AG_1 - xG_2)}{A}} \quad (4.5)$$

$$= \frac{AG_1 - pG_2}{AG_1 - xG_2} \quad (4.6)$$

For the special case $p > 0$, $x = 0$ we have:

$$\Psi(p, 0) = \frac{AG_1 - pG_2}{A}. \quad (4.7)$$

Using (4.7) it is not difficult to show that (4.2) holds for $p \leq x$ and holds for $p \geq x$ as well. Formula (4.7) easily invertible, for

$$\frac{1}{A} = \frac{2\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \frac{s\Gamma\left(\frac{s+1}{2}\right)}{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)} = \frac{2\Gamma\left(\frac{s+1}{2}\right)}{\frac{s}{2}\Gamma\left(\frac{s}{2}+1\right)}.$$

Clearly $\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}$ is the Laplace transform of $2 \cdot \frac{e^{-t}}{\sqrt{\pi(1-e^{-2t})}}$. Hence (4.7) is the

Laplace transform of:

$$2 \cdot \frac{d}{dt} \left\{ \frac{e^{-\frac{p^2 e^{-2t}}{(1-e^{-2t})}}}{\sqrt{\pi(1-e^{-2t})}} * \frac{e^{-t}}{\sqrt{\pi(1-e^{-2t})}} \right\} = 2 \cdot \frac{d}{dt} \frac{1}{\sqrt{\pi}} \int_{\frac{pe^{-t}}{\sqrt{1-e^{-2t}}}}^{\infty} e^{-z^2} dz = \frac{2pe^{-t}}{\sqrt{\pi}} \frac{e^{-\frac{p^2 e^{-2t}}{1-e^{-2t}}}}{(1-e^{-2t})^{\frac{3}{2}}}.$$

5 The Equivalence of Bellman-Haris' and our Result

To show that formulas (4.3) and (4.4) are the same, we have to evaluate the

$$\text{integral } \int_0^{\infty} e^{-y^2+2xy} dy = e^{x^2} \cdot e^{-(x-y)^2} = e^{x^2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \frac{d^n e^{-y^2}}{dy^n}.$$

Substituting this into the integral we get:

$$\int_0^{\infty} e^{-y^2+2xy} y^{s-1} dy = e^{x^2} \cdot \int_0^{\infty} e^{-(x-y)^2} y^{s-1} dy = e^{x^2} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \int_0^{\infty} \frac{d^n e^{-y^2}}{dy^n} y^{s-1} dy. \quad (5.1)$$

Let us observe that the integrals on the right hand side are the Mellin transforms of

the functions $\frac{d^n e^{-y^2}}{dy^n}$. First we compute the Mellin transform of e^{-y^2} which is:

$$\int_0^{\infty} e^{-y^2} y^{s-1} dy = \frac{1}{2} \int_0^{\infty} e^{-y} y^{\frac{s}{2}-1} dy = \frac{1}{2} \Gamma\left(\frac{s}{2}\right).$$

Let us denote the Mellin transform of a function f by $\mathcal{M}f$ or F . It is not difficult to see that:

$$\begin{aligned}\mathcal{M}(f') &= -(s-1)F(s-1) \\ \mathcal{M}(f'') &= -(s-1)(s-2)F(s-2) \\ &\dots \\ &\dots \\ \mathcal{M}(f^n) &= (-1)^n (s-1)(s-2)\dots(s-n)F(s-n).\end{aligned}$$

Hence the Mellin transforms of e^{-y^2} , $\frac{de^{-y^2}}{dy}$, $\frac{d^2e^{-y^2}}{dy^2}$, $\frac{d^3e^{-y^2}}{dy^3}$... are $\frac{1}{2}\Gamma\left(\frac{s}{2}\right)$, $-\frac{(s-1)}{2}\Gamma\left(\frac{s-1}{2}\right)$, $\frac{(s-2)(s-1)}{2}\Gamma\left(\frac{s-2}{2}\right)$, $\frac{(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-3}{2}\right)$, $\frac{(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-4}{2}\right)$, $\frac{(s-5)(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-5}{2}\right)$, ...

Substituting these into (5.1) gives:

$$\begin{aligned}& e^{x^2} \left\{ \frac{1}{2}\Gamma\left(\frac{s}{2}\right) + \frac{x}{1!} \frac{s-1}{2}\Gamma\left(\frac{s-1}{2}\right) + \frac{x^2}{2!} \frac{(s-2)(s-1)}{2}\Gamma\left(\frac{s-2}{2}\right) \right. \\ & + \frac{x^3}{3!} \frac{(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-3}{2}\right) + \frac{x^4}{4!} \frac{(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-4}{2}\right) \\ & \left. + \frac{x^5}{5!} \frac{(s-5)(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-5}{2}\right) + \dots \right\} \\ & = e^{x^2} \left\{ \frac{1}{2}\Gamma\left(\frac{s}{2}\right) + \frac{x^2}{2!} \frac{(s-2)(s-1)}{2}\Gamma\left(\frac{s-2}{2}\right) \right. \\ & \left. + \frac{x^4}{4!} \frac{(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-4}{2}\right) + \dots \right\} \\ & + e^{x^2} \left\{ \frac{x}{1!} \frac{s-1}{2}\Gamma\left(\frac{s-1}{2}\right) + \frac{x^3}{3!} \frac{(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-3}{2}\right) \right. \\ & \left. + \frac{x^5}{5!} \frac{(s-5)(s-4)(s-3)(s-2)(s-1)}{2}\Gamma\left(\frac{s-5}{2}\right) + \dots \right\} \\ & = e^{x^2} \left\{ \frac{1}{2}\Gamma\left(\frac{s}{2}\right) + \frac{x^2}{2!} 2^0 \Gamma(s-1) \Gamma\left(\frac{s}{2}\right) + \frac{x^4}{4!} 2^1 (s-3)(s-1) \Gamma\left(\frac{s}{2}\right) + \dots \right\}\end{aligned}$$

$$\begin{aligned}
& + e^{x^2} \left\{ \frac{x}{1!} 2^0 \Gamma\left(\frac{s+1}{2}\right) + \frac{x^3}{3!} 2^1 (s-2) \Gamma\left(\frac{s+1}{2}\right) \right. \\
& \left. + \frac{x^5}{5!} 2^2 (s-4)(s-2)(s-1) \Gamma\left(\frac{s+1}{2}\right) + \dots \right\} \\
& = e^{x^2} \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \left\{ 1 - \frac{x^2}{1!} \frac{\frac{1-s}{2}}{\frac{1}{2}} + \frac{x^4}{2!} \frac{\frac{1-s}{2}}{\frac{1}{2}} \frac{\frac{3-s}{2}}{\frac{3}{2}} + \dots \right\} \\
& + e^{x^2} \Gamma\left(\frac{s+1}{2}\right) x \left\{ 1 - \frac{x^2}{1!} \frac{\frac{1-s}{2}}{\frac{3}{2}} + \frac{x^4}{2!} \frac{\frac{1-s}{2}}{\frac{3}{2}} \frac{\frac{2-s}{2}}{\frac{5}{2}} + \dots \right\} \\
& = e^{x^2} \frac{1}{2} \Gamma\left(\frac{s}{2}\right) F\left(\frac{1-s}{2}; \frac{1}{2}, -x^2\right) + e^{x^2} \Gamma\left(\frac{s+1}{2}\right) x F\left(1 - \frac{s}{2}, \frac{3}{2}; -x^2\right) \\
& = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) F\left(\frac{s}{2}, \frac{1}{2}; -x^2\right) + \Gamma\left(\frac{s+1}{2}\right) x F\left(\frac{1+s}{2}, \frac{3}{2}; x^2\right).
\end{aligned}$$

Hence Bellman and Harris formula becomes:

$$\frac{\frac{1}{2} \Gamma\left(\frac{s}{2}\right)}{\int_0^\infty e^{-y^2+2xy} y^{s-1} dy} = \frac{\frac{1}{2} \Gamma\left(\frac{s}{2}\right)}{\frac{1}{2} \Gamma\left(\frac{s}{2}\right) F\left(\frac{s}{2}, \frac{1}{2}; x^2\right) + \Gamma\left(\frac{s+1}{2}\right) x F\left(\frac{1+s}{2}, \frac{3}{2}; x^2\right)}.$$

Diving both the numerator and the denominator of the right hand side into

$\Gamma\left(\frac{s+1}{2}\right)$ gives $\frac{A}{AG_1 + xG_2}$ and this completes the proof.

Reference

- [1] Bellman, R., Harris, T.: Recurrence times for the Ehrenfest model. Pacific J. Math. 1. 179-193 (1951)