Fuzzy Linear Systems Applied to Leontief Input-Output Model

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Abstract

A general fuzzy linear system is investigated using fuzzy numbers and Gauss-Seidel iteration formula. We have used our fuzzy linear system to solve Leontief input-output model with fuzzy entries. When solving Leontief input-output model one is usually making the assumption that we know entirely the consumption matrix from industrial entries and we are certain about the final demand. These assumptions however depend heavily on estimates and information received from the industry and hence in these estimates, uncertainty plays a crucial role. To address this type of uncertainty fuzzy methods are needed to model this and in this article we are giving a procedure to solve this problem. Numerical examples are given to illustrate the procedure. Among them also the famous example from Leontief himself where he solved the production levels for U.S. economy in 1958.

Key words: Fuzzy Linear System, Leontief input-output model, fuzzy numbers, Gauss-Seidel, SOR

1 Introduction

Linear systems played an essential role in the Nobel prize-winning work of Wassily Leontief. The economic model described by him is the basis for more elaborate models now in many parts of the world.

We introduce the topic by basing it on David C. Lay's text book [1]. Suppose the nation's economy is divided into n sectors that produce goods or services, and let $\mathbf{x}$ be a *production vector* in $\mathbb{R}^n$ that lists the output of each sector for one year. Also, suppose another part of the economy (called the *open sector*) does not produce goods or services but only consumes them, and let $\mathbf{d}$ be a *final demand vector* (or *bill of final demands*) that lists the value of the goods and services demanded from the various sectors by the nonproductive part of the economy. The vector $\mathbf{d}$ can represent consumer demand, government consumption, surplus production, exports, or other external demand.

As the various sectors produce goods to meet consumer demand, the produces themselves create additional *intermediate demand* $\mathbf{i}$ for goods they need as inputs for their own production. The interrelations between the sectors is very complex, and the connection between the final demand and the production is unclear. Leontief asked if there is a production level $\mathbf{x}$ such that the amounts produced (or "supplied") will exactly balance the total demand for that production, so that

$$\mathbf{x} = \mathbf{i} + \mathbf{d} \tag{1}$$

The basic assumption of Leontief's input-output model is that for each sector, there is a *unit consumption vector* in $\mathbb{R}^n$ that lists the inputs needed *per unit of output* of the sector. All input and output units are measured in millions of dollars, rather that in quantities such as tons or bushels. Prices of goods and services are held constant.

Example 1.1 Suppose the economy consists of three sectors – manufacturing, agriculture, and services – with unit consumption vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ shown in the table below:

Purchased from	Manufacturing	Agriculture	Services
Manufacturing	0.50	0.40	0.20
Agriculture	0.20	0.30	0.10
Services	0.10	0.10	0.30

The columns of manufacturing, agriculture, and services consist of inputs consumed per unit of output. The manufacturing column is $\mathbf{c}_1$, the agriculture column is $\mathbf{c}_2$, and the services column is $\mathbf{c}_3$. What amounts will be consumed by the manufacturing sector if it decides to produce 100 units?

For the solution, compute

$$100\mathbf{c}_1 = 100 \begin{pmatrix} 0.50 \\ 0.20 \\ 0.10 \end{pmatrix} = \begin{pmatrix} 50 \\ 20 \\ 10 \end{pmatrix}$$

To produce 100 units, manufacturing will order (i.e., "demand") and consume 50 units from other parts of the manufacturing sector, 20 units from agriculture, and 10 units from services.

2 Leontief input-output model

Input-output tables (I/O-table, for short) form a coherent frame for keeping of accounts for describing itinerant commodity floods in national economy. From the I/O-table, the *establishing balance equation*

$$x_i = x_{i1} + x_{i2} + \ldots + x_{in} + y_i \quad (i = 1, \ldots, n) \tag{2}$$

appears. This shows that the whole production x_i of the each sector is used as intermediate inputs $x_{i1}, x_{i2}, \ldots, x_{in}$ in the sectors $1, 2, \ldots, n$ and partly to the final output y_i. (Notice that in some sector i, x_{ii} may differ from zero, the all other x_{ij}'s may be equal to zero, or y_i may be equal to zero.)

The *input coefficients* are

$$a_{ij} = \frac{x_{ij}}{x_j} \tag{3}$$

where x_{ij} stands for the use of products from a sector i as input in a sector j and x_j stands for the total production in a sector j.

From the input coefficients a_{ij} we form the *consumption matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{4}$$

The *basic assumption* of the input-output model is that the input coefficients are fixed, i.e., the "recipe" of the model is supposed to keep constant regardless of the production amount. Using input coefficients, the establishing balance equation (2) can be put into the form

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + y_i \quad (i = 1, \ldots, n). \tag{5}$$

These equations can be put into the matrix form

$$X = AX + Y \tag{6}$$

where

$$X = (x_1, \ldots, x_n)^{\mathrm{T}} \quad \text{and} \quad Y = (y_1, \ldots, y_n)^{\mathrm{T}}.$$

Input-output model can be used in several tasks. Some of them are

(a) to determine total production of each sectors and transactions between them if the final demand is known,

(b) to determine the need of basic inputs if the final demand is known,
(c) to analyze the relational change of prices and to find out the expense construction.

In the case (a), we want to find out all accumulating influences, if a certain amount of products is produced for final demand (producing bread needs corn, producing corn needs tractors, producing tractors needs metal, workers of metal industry need bread etc.). Mathematically, this means that Y is known in the equation (6) and we want to calculate X. The solution can be found as follows. First we put (6) into the form

$$Y = X - AX = (I - A)X, \tag{7}$$

from which we have

$$X = (I - A)^{-1}Y. \tag{8}$$

Substituting the vector $Y = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ to (7) where the jth component equals to 1 and other components are equal to zero, we see that the element $\widehat{a}_{kj}$ of the matrix $(I - A)^{-1}$ has a relevant interpretation. The element $\widehat{a}_{kj}$ expresses how much together we need to produce, taking into consideration all the accumulating influences, in order to produce one unit for final demand in the sector j.

Example 2.1 *Determine the total demand X for industry sectors* 1, 2, *and* 3 *if*

$$A = \begin{pmatrix} 0,3 & 0,4 & 0,1 \\ 0,5 & 0,2 & 0,6 \\ 0,1 & 0,3 & 0,1 \end{pmatrix} \quad \textit{and} \quad Y = \begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix}.$$

Hence, by (6) we have

$$\begin{gathered} X - AX = Y \iff (I - A)X = Y \iff \\ X = (I - A)^{-1}Y \implies \end{gathered} \tag{9}$$

$$\begin{aligned} (I - A) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0,3 & 0,4 & 0,1 \\ 0,5 & 0,2 & 0,6 \\ 0,1 & 0,3 & 0,1 \end{pmatrix} \\ &= \begin{pmatrix} 0,7 & -0,4 & -0,1 \\ -0,5 & 0,8 & -0,6 \\ -0,1 & -0,3 & 0,9 \end{pmatrix} \implies \end{aligned} \tag{10}$$

$$(I-A)^{-1} = \frac{1}{0,151}\begin{pmatrix} 0,54 & 0,39 & 0,32 \\ 0,51 & 0,62 & 0,47 \\ 0,23 & 0,25 & 0,36 \end{pmatrix} \Longrightarrow$$

$$X = \frac{1}{0,151}\begin{pmatrix} 0,54 & 0,39 & 0,32 \\ 0,51 & 0,62 & 0,47 \\ 0,23 & 0,25 & 0,36 \end{pmatrix}\begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 160,93 \\ 201,99 \\ 118,54 \end{pmatrix}.$$

The result is $x_1 = 160,93$, $x_2 = 201,99$ *and* $x_3 = 118,54$.

In the case (b), the need of basic inputs, when the final demand is known, is considered as follows. First, to find out the total production in different sectors, we proceed in the similar way as is done in the case (a). After this, the *coefficient matrix of basic inputs* created in the same way as the matrix A is used. A *basic input coefficient* is defined as

$$d_{ij} = \frac{z_{ij}}{x_j} \tag{11}$$

where z_{ij} stands for the use of the basic input i in the sector j and x_j stands for the total demand of the sector j. Hence, the coefficient matrix of basic inputs is

$$D = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{pmatrix}. \tag{12}$$

If Z is a basic input vector then we have

$$Z = D(I-A)^{-1}Y. \tag{13}$$

Also here, an element $\widehat{d}_{kj}$ of the matrix $D(I-A)^{-1}$ has a relevant interpretation. The element $\widehat{d}_{kj}$ indicates the total amount of the basic input k in order to produce one unit for the final demand j.

In the case (c), it is thought that the price of each product is composed of the costs of used intermediate inputs and basic inputs. The price equation now corresponding to establishing balance equation (2) is

$$p_j = a_{1j}p_1 + a_{2j}p_2 + \ldots + a_{nj}p_n + w_j \qquad (j = 1, \ldots, n), \tag{14}$$

where p_j is the unit price of production of the sector j and w_j is the unit costs of basic inputs of the sector j. The equation (14) in matrix form is

$$P = A^{\mathrm{T}}P + W. \tag{15}$$

If we suppose that W is known, P then has the form

$$P = (I - A^{\mathrm{T}})^{-1}W. \tag{16}$$

The model we just introduced is crisp. As we noticed above, the basic assumption of the model is that the input coefficients are fixed and hence, the core of model is constant. One reason for this is, that the matrix A, is in practice very big. But this assumption causes sometimes some troubles. To avoid them, one way may be to fuzzify A. So, the rest of the paper considers possible fuzzy tools for solving the problem by constructing a fuzzy model where the matrix A consists of fuzzy numbers.

3 Fuzzy Linear Systems

Fuzzy linear systems can occur in several research fields, e.g. in control problems, statistics, physics, engineering, information, economics, and finance science. In [2] following fuzzy linear system (FLS) was considered,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{cases} \tag{17}$$

where their coefficient matrix $A = (a_{ij})$ was considered to be crisp matrix and $\mathrm{y} = (\mathrm{y_i})$ was a fuzzy vector, $1 \leq i, j \leq n$. They solved this type of fuzzy system using embedding method. In this article our fuzzy linear system differs from [2] so that we consider the coefficient matrix $A = (a_{ij})$ to be fuzzy coefficients also instead of just y vector to be fuzzy. We solve this problem by using arithmetic operations for left-right(LR)-type fuzzy numbers and apply them to Gauss-Seidel algorithm. Based on Friedman et al [2] work many numerical methods [3–13] have been presented for this type of FLS. In this article we present a way to solve FLS for the case where also coefficient matrix $A = (a_{ij})$ is considered to be fuzzy. The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [14,15]. The Gauss-Seidel algorithm was first applied to FLS with crips coefficient matrix A in [5]. This method was modified to cover Successive Over Relaxation (SOR) method [6] later. In this article also coefficient matrix A is fuzzy and we apply Gauss-Seidel algorithm to solve our FLS and use LR-type fuzzy numbers and arithmetic operations to them introduce by Dubois & Prade [16]. We also present how this can be modified to Successive Over Relaxation (SOR) method. We apply our method to solve Leontief input-output model.

We are solving

$$AX = Y$$

where A is matrix and X and Y are vectors. We assume that instead of A and Y being crips they are considered to consist of fuzzy numbers. We use left-right-type fuzzy numbers and Gauss-Seidel method to solve X for fuzzy numbers. This way we also receive fuzzy support area for X.

Gauss-seidel iteration formula is

$$x_i^{k+1} = \frac{1}{a_{ii}} \left(y_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{N} a_{ij} x_j^k \right) \tag{18}$$

$i = 1, ..., N, a_{ii} \neq 0$. We are using Gauss-seidel iteration because with this we can simply use extented addition, difference, extented product and extented quotient introduced in [16] which are relatively straightforward to implement. We are using left-right type fuzzy numbers, which can be written as

$$\mu_A(x) = \begin{cases} L((M-x)/l) \text{ if } x \leq M \\ R((x-M)/r) \text{ if } x \geq M \end{cases} \tag{19}$$

for $l > 0, r > 0$. For L-R-type function we used $L(x) = R(x) = max\{1-x, 0\}$.

We used following extensions: For fuzzy number $A_1 = (M_1, l_1, r_1)_{LR}$, M_1 is the modal value, l_1 is left support length and r_1 is right support length. and same for $A_2 = (M_2, l_2, r_2)_{LR}$.

Extented sum:

$$A_1 \oplus A_2 = (M_1 + M_2, l_1 + l_2, r_1 + r_2)_{LR} \tag{20}$$

Extented difference:

$$A_1 \ominus A_2 = (M_1 - M_2, l_1 + r_2, r_1 + l_2)_{LR} \tag{21}$$

The extented product:

When $M_1 > 0$ and $M_2 > 0$

$$A_1 \odot A_2 \approx (M_1 M_2, M_1 l_2 + M_2 l_1, M_1 r_2 + M_2 r_1)_{LR} \tag{22}$$

When $M_1 < 0$ and $M_2 > 0$

$$A_1 \odot A_2 \approx (M_1 M_2, M_2 l_1 - M_1 r_2, M_2 r_1 - M_1 l_2)_{RL} \tag{23}$$

if spreads are small w.r.t. M_1 and M_2, and instead of equation (22)

$$A_1 \odot A_2 \approx (M_1 M_2, M_1 l_2 + M_2 l_1 - l_1 l_2, M_1 r_2 + M_2 r_1 - r_1 r_2)_{LR} \tag{24}$$

should be used if spreads are not small w.r.t. M_1 and M_2. Similarly with equation (23), see more about these product rules e.g. in Dubois and Prades book [16]. Notice that we do not need to deal with cases when $M_2 < 0$ when solving production levels using Leontief's I/O model with fuzzy entries since production levels can not be negative. This basically means that when solving $AX = Y$, X which is production level vector we can not have negative production level values.

The extented quotient can be expressed as

$$A_1 \oslash A_2 \approx (\frac{M_1}{M_2}, \frac{r_2 M_1 + l_1 M_2}{M_2^2}, \frac{l_2 M_1 + r_1 M_2}{M_2^2})_{LR} \tag{25}$$

Using these LR-type fuzzy operations our fuzzy Gauss-Seidel iteration formula looks like

$$x_i^{k+1} = \left(y_i \ominus \left(\oplus_{j=1}^{i-1} a_{ij} \odot x_j^{k+1}\right) \ominus \left(\oplus_{j=i+1}^{N} a_{ij} \odot x_j^{k}\right)\right) \oslash a_{ii} \tag{26}$$

Usually $||\cdot||_\infty$ is used for the ending criterion to iterations. Here we also apply $\infty-$norm but now we extended it so that it also takes into account the support areas. So $||X^{k+1} - X^k||_\infty < \epsilon$ is now extented to $max\{||M^{k+1} - M^k||_\infty, \quad ||l^{k+1} - l^k||_\infty, ||r^{k+1} - r^k||_\infty\} < \epsilon$. This change guarantees that also support areas are converged. This Gauss-Seidel iteration formula was implemented using Matlab^{TM}

Modification to Successive Over Relaxation (SOR) method

Gauss-Seidel method can be modified to relaxation parameter. Modifying Gauss-Seidel method to include this relaxation parameter is often also called as Successive Over Relaxation (SOR) method [17]. When applying Gauss-Seidel method the components for the change $\epsilon = x^{k+1} - x^k$ are usually with same sign without depending on iteration number k. In this case one can accelerate the iteration process so that instead of giving the result from Gauss-Seidel iteration

$$\tilde{x}_i^{k+1} = \frac{1}{a_{ii}} \left(y_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{N} a_{ij} x_j^k \right) \tag{27}$$

the iteration result is given in

$$x_i^{k+1} = x_i^k + \omega(\tilde{x}_i^{k+1} - x_i^k) \tag{28}$$

where the normal result $\epsilon_i = \tilde{x}_i^{k+1} - x_i^k$, which gives the result $x_i^{k+1} = x_i^k + \epsilon_i = \tilde{x}_i^{k+1}$, is enhanced by coefficient ω. Value $\omega = 1$ gives the usual Gauss-Seidel method. When $\omega < 1$, we have under relaxation and when $\omega > 1$ we have over relaxation. This relaxation method converges when $\omega \in (0, 2)$. Optimal ω is here found by trial. As a computational formula for this method we get

$$x_i^{k+1} = (1-\omega) x_i^k + \frac{\omega}{a_{ii}} \left(y_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{N} a_{ij} x_j^k \right) \tag{29}$$

Now when using above mentioned fuzzy arithmetics, besided previously introduced formulas we also need scalar multiplication which is defined as

$$\omega A_1 = (\omega M_1, |\omega| l_1, |\omega| r_1)_{LR} \tag{30}$$

and now fuzzy SOR iteration formula is

$$x_i^{k+1} = (1-\omega) x_i^k \oplus \omega \left(y_i \ominus \left(\oplus_{j=1}^{i-1} a_{ij} \odot x_j^{k+1} \right) \ominus \left(\oplus_{j=i+1}^{N} a_{ij} \odot x_j^k \right) \right) \oslash a_{ii} \tag{31}$$

Now notice that since scalar multiplication is defined as above we suggest that relaxation parameter is now chosen so that $\omega \in (0, 1]$.

4 Results

We illustrate the performance of our method using three different examples.

Example 1: Consider the economy whose consumption matrix is given as

$$C = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$$

Suppose the final demand is 50 units for manufacturing, 30 units for agriculture, and 20 units for services. Assume that these entries are fuzzy numbers and left and right fuzzy support length equaling $l = r = 0.01$. Next we need to find the production level x that will satisfy this demand. In Table 1 there are results from this experiment with crisp case and also with fuzzy entries. In Figure 1 we have also plotted the membership values for our fuzzy production level.

Table 1
Production levels solve with crisp solution in left and solution with fuzzy entries on the right. Solution for fuzzy entries is given in form X=[M,l,r] where M is modal value and l and r are left and right support.

X **crisp**	X **fuzzy**
225.93	[225.93 25.08 25.08]
118.51	[118.52 14.89 14.89]
77.78	[77.78 11.76 11.76]

Example 2: Steel manufacturer, who controlled 30% of the markets, wants to investigate how the change in demand for car industry effects the demand in steel manufacturing industry. To simplify the example only car industry, steel industry, railroad industry and mining industry is considered. We assumme that entries (A) were calculated for this year and are given in Table 2. Demand for the end product was this year $Y_1 = (10, 100, 30, 10)$. For the next year it was predicted that demand for cars would grow by 20% and demand for others would stay the same, so $Y_2 = (10, 120, 30, 10)$. Needed production levels can then be solved by $X_i = (I - A)^{-1}Y_i, \quad i = 1, 2.$ where I is identity matrix.

Solving this for crisp case for both years we received results reported in Table 3.

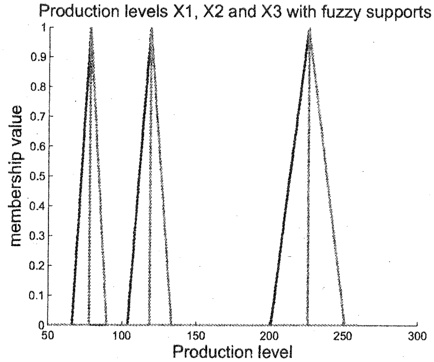

Fig. 1. Fuzzy production levels in example 1, when consumption matrix and final demand is considered to be fuzzy.

Table 2
Industries entries

	steel industry	car industry	railroad industry	mining industry
steel industry	0.1	0.4	0.1	0.1
car industry	0	0.1	0.4	0.2
railroad ind.	0.2	0.1	0	0.2
mining ind.	0.3	0	0	0

Table 3
Production levels for this year and predictions for next

	X_1	X_2
steel industry	90.0	102.3
car industry	150.7	176.0
railroad industry	70.4	76.0
mining industry	37.0	40.7

Next we solved this with fuzzy matrix A and vector Y with left and right fuzzy support length equaling $l = r = 0.05$ for all (a_{ij}) values in fuzzy matrix A. We received results reported in Table 4. Results are given so that first modal value is given then left support and then right support. Results are also plotted in Figure 2 a and 2 b. In Figure 2 c we calculated the differences in production levels for these two years. As can be seen from the figure uncertainties in the predictions can effect quite much the final production level growth.

As we can see the modal values are about the same as in crisp cases. What is notable is how much the support areas in matrix A are influencing the support areas of our solution X. This clearly shows that one needs to take into account the uncertainty that can exists in A and how they are influencing the solution. To study

Table 4
Production levels for this year and predictions for next year with fuzzy supports

	X_1	X_2
steel industry	[90.0 11.6 11.6]	[102.5 51.5 51.5]
car industry	[150.7 11.3 11.3]	[176.3 44.8 44.8]
railroad industry	[70.5 11.2 11.2]	[76.3 41.2 41.2]
mining industry	[37.0 14.0 14.0]	[40.7 33.2 33.2]

Table 5
Differences in spreads if spread is double in car industry sector

	$X2_{orig}$	$X2_{double}$
steel industry	[102.5 51.5 51.5]	[102.5 60.2 60.2]
car industry	[176.3 44.8 44.8]	[176.3 65.5 65.5]
railroad industry	[76.3 41.2 41.2]	[76.3 45.5 45.5]
mining industry	[40.7 33.2 33.2]	[40.7 35.6 35.6]

more about how different support areas are influencing the results we decided to double the support area to all a_{ij} values for one row and keep everything else as in previous experiment. In Table 5 one can see how doubling the support area in car industry and keeping the support areas the same for other industries influences the results. In second column is the original results and in third column the results from this experiment. These results are also plotted in Figure 2 d. As can be seen support areas are growing in all cases when fuzziness is increasing but much less for railroad industry and mining industry than car industry.

As our last example to demonstrate how our method can be applied in Leontief's input output model we consider the famous case, which Leontief solved for U.S. economy data in 1958. This time of course, we do it with fuzzy case.

Example 3: The consumption matrix C below is based on input-output data for the U.S. economy in 1958, with data for 81 sectors grouped into 7 larger sectors: (1) nonmetal household and personal products, (2) final metal products (such as motor vehicles), (3) basic metal products and mining, (4) basic nonmetal products and agriculture, (5) energy (6) services, and (7) entertainment and miscellaneous products [18]. We need to find the production levels to satisfy the final demand y_1. (Units are in millions of dollars.)

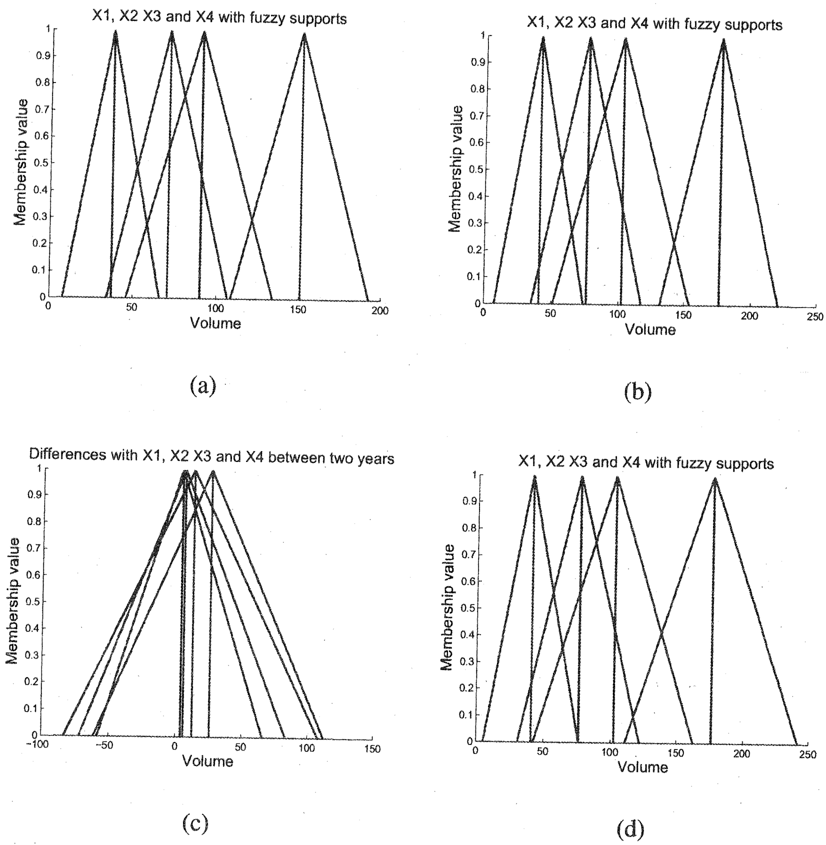

Fig. 2. Production levels with fuzzy spreads a) for year 1, b)predictions for the year 2, c) differences between predictions for next year and production levels for this year, and d) predictions for the year 2 when car industry is considered to have double amount the uncertainty than in others.

$$C = \begin{bmatrix} 0.1588 & 0.0064 & 0.0025 & 0.0304 & 0.0014 & 0.0083 & 0.1594 \\ 0.0057 & 0.2645 & 0.0436 & 0.0099 & 0.0083 & 0.0201 & 0.3413 \\ 0.0264 & 0.1506 & 0.3557 & 0.0139 & 0.0142 & 0.0070 & 0.0236 \\ 0.3299 & 0.0565 & 0.0495 & 0.3636 & 0.0204 & 0.0483 & 0.0649 \\ 0.0089 & 0.0081 & 0.0333 & 0.0295 & 0.3412 & 0.0237 & 0.0020 \\ 0.1190 & 0.0901 & 0.0996 & 0.1260 & 0.1722 & 0.2368 & 0.3369 \\ 0.0063 & 0.0126 & 0.0196 & 0.0098 & 0.0064 & 0.0132 & 0.0012 \end{bmatrix}$$

$$y_1 = [74000 \quad 56000 \quad 10500 \quad 25000 \quad 17500 \quad 196000 \quad 5000].$$

Table 6
Production levels for this year and predictions for next year with fuzzy supports

X_{1958} crisp	X_{1958} with fuzzy entries	X_{1964}	X_{1964} with fuzzy entries
99576	[99576 6151 6151]	134034	[134034 8283 8283]
97703	[97703 8584 8584]	131686	[131686 11560 11560]
51231	[51231 9012 9012]	69472	[69472 12136 12136]
131570	[131570 12494 12494]	176912	[176912 16825 16825]
49488	[49488 7587 7587]	66596	[66596 10217 10217]
329554	[329554 13995 13995]	443773	[443773 18847 18847]
13835	[13835 4550 4550]	18431	[18431 6127 6127]

We solve production levels for this problem using left and right support values as $l = r = 0.005$. In Table 6 are the production levels solved with fuzzy I/O-model. Leontief also solved production levels for year 1964 using the same consuption matrix C with demand vector y_2 now being

$$y_2 = [99640 \quad 75548 \quad 14444 \quad 33501 \quad 23527 \quad 263985 \quad 6526].$$

We repeated this experiment with out fuzzy entries and the solution with this demand vector is also given in Table 6. As can be seen we calculated the production levels and predicted production levels for U.S. economy and take the uncertainty involved, quite easily with our proposed method.

5 Discussion

In this article we applied Gauss-Seidel iterative method and SOR interative method to approximate of the unique solution for fuzzy linear system. We have applied this solution to solve Leontief input-output model. This seems practical and gives valuable information since in the cases presented here, the consumption matrix is inherently fuzzy. This is especially true in cases where it is used to approximated production levels for the next year as we have demonstrated in our examples. We can even calculate approximations going further than just next year as our last example showed. Here also the final demand is clearly fuzzy. Using fuzzy I/O-model we managed to approximate the production levels needed for production in the year in question and also estimate production levels for year to come and succesfully calculated the support area when the consumption matrix and final demand was considered to be fuzzy numbers.

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