Youla-Parametrized Controllers for Stable Multivariable Processes

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Abstract: The *Youla-parametrization* is a useful method to design controllers for open-loop stable plants. The *KB-parametrization* is a successful extension of this method for two-degree-of freedom (*2DOF*) systems. The paper extends this methodology for multivariable case.

Keywords: Youla-parametrization, multivariable process, MIMO controller

1. Introduction to Multivariable Process Models

Consider a *M*ulti-*I*nput-*M*ulti-*O*utput (*MIMO*) continuous time *L*inear *T*ime *I*nvariant (*LTI*) dynamic plant described by the state variable representation (*SVR*)

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{t}} &= \dot{\boldsymbol{x}} = \mathbf{A}\,\boldsymbol{x} + \mathbf{B}\,\boldsymbol{u} \\ \boldsymbol{y} &= \mathbf{C}^{\mathrm{T}}\boldsymbol{x} + \mathbf{D}\,\boldsymbol{u} \end{aligned} \tag{1}$$

Here $\boldsymbol{u}$, $\boldsymbol{y}$ and $\boldsymbol{x}$ are the input, output and state variables of the process to be controlled and $^{\mathrm{T}}$ stands for transposition.

The transfer function representation (*TFR*) of the *MIMO* plant can be calculated by

$$\mathbf{P}(s) = \mathbf{C}^{\mathrm{T}}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} = \mathbf{C}^{\mathrm{T}}\boldsymbol{\Phi}(s)\mathbf{B}+\mathbf{D} = \frac{1}{\mathcal{A}(s)}\boldsymbol{\mathcal{B}}(s) \tag{2}$$

where

$$\boldsymbol{\Phi}(s) = (s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{\mathcal{A}(s)}\boldsymbol{\Psi}(s) \quad ; \quad \boldsymbol{\Psi}(s) = \mathbf{adj}(s\mathbf{I}-\mathbf{A}) \tag{3}$$

and the scalar denominator polynomial

$$\mathcal{A}(s) = \det(s\mathbf{I}-\mathbf{A}) \tag{4}$$

is the characteristic polynomial. Assume that the process is square, i.e. the number of input and output variables is the same: p. So $\mathbf{C}$ is a $p \times n$, $\mathbf{B}$ is an $n \times p$, $\mathbf{D}$ is a $p \times p$, $\mathbf{A}$ is an $n \times n$ constant parameter matrix, furthermore $\mathbf{I}$ is the $n \times n$ unit matrix. Consequently the degree of $\mathcal{A}(s)$ is also n. The $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are $n \times n$ polynomial matrices, $\boldsymbol{P}(s)$ is an $n \times n$ transfer function matrix (TFM).

The formal "numerator" matrix polynomial

$$\mathcal{B}(s) = \mathbf{C}^{\mathrm{T}}\boldsymbol{\Psi}(s)\mathbf{B} + \mathcal{A}(s)\mathbf{D}\Big|_{\mathbf{D}=0} = \mathbf{C}^{\mathrm{T}}\boldsymbol{\Psi}(s)\mathbf{B} \tag{5}$$

is commutative with the scalar denominator $\mathcal{A}(s)$. So the form (2) of $\boldsymbol{P}(s)$ is the simplest *MIMO* model, which is not irreducible, however does not involve the more complex forms of the left and right *MFD*'s (*M*atrix *F*ractional *D*escription, see later). It is sometimes called "naive" model representation.

A *MIMO* TFM $\boldsymbol{P}(s)$ can be transformed into the so-called SMITH-form

$$\boldsymbol{P}(s) = \boldsymbol{\mathcal{U}}(s)\boldsymbol{D}(s)\boldsymbol{\mathcal{V}}(s) \tag{6}$$

where $\boldsymbol{\mathcal{U}}(s)$ and $\boldsymbol{\mathcal{V}}(s)$ are unimodular polynomial matrices. (A matrix is unimodular, if its determinant is a scalar constant. The inverses of unimodular matrices are also polynomial ones.) The matrix $\boldsymbol{D}(s)$ is a special diagonal TFM

$$\begin{aligned}\boldsymbol{D}(s) &= \boldsymbol{D}_{\mathrm{D}}(s)\boldsymbol{\mathcal{D}}_{\mathrm{N}}(s) = \boldsymbol{\mathcal{D}}_{\mathrm{N}}(s)\boldsymbol{D}_{\mathrm{D}}(s) = \boldsymbol{\mathcal{D}}_{\mathrm{D}}^{-1}(s)\boldsymbol{\mathcal{D}}_{\mathrm{N}}(s) = \boldsymbol{\mathcal{D}}_{\mathrm{N}}(s)\boldsymbol{\mathcal{D}}_{\mathrm{D}}^{-1}(s) = \\ &= \mathbf{diag}\left\langle \frac{\varepsilon_1}{\psi_1}, \frac{\varepsilon_2}{\psi_2}, \ldots, \frac{\varepsilon_{\mathrm{r}}}{\psi_{\mathrm{r}}}, 0, \ldots, 0 \right\rangle = \mathbf{diag}\left\langle \lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{r}}, 0, \ldots, 0 \right\rangle\end{aligned} \tag{7}$$

where the diagonal matrices are commutative. Here

$$\boldsymbol{\mathcal{D}}_{\mathrm{N}}(s) = \mathbf{diag}\left\langle \varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_{\mathrm{r}}(s), 0, \ldots, 0 \right\rangle ; \lambda_i(s) = \frac{\varepsilon_i(s)}{\psi_i(s)} \tag{8}$$

and

$$\boldsymbol{D}_{\mathrm{D}}(s) = \boldsymbol{\mathcal{D}}_{\mathrm{D}}^{-1}(s) \quad ; \quad \boldsymbol{\mathcal{D}}_{\mathrm{D}}(s) = \mathbf{diag}\left\langle \psi_1(s), \psi_2(s), \ldots, \psi_{\mathrm{r}}(s), 0, \ldots, 0 \right\rangle \tag{9}$$

The $\varepsilon_i(s)$ and $\psi_i(s)$ are coprime polynomials. The diagonal $\boldsymbol{D}(s)$ TFM can be expressed from $\boldsymbol{P}(s)$ as

$$\boldsymbol{D}(s) = \boldsymbol{\mathcal{U}}^{-1}(s)\boldsymbol{P}(s)\boldsymbol{\mathcal{V}}^{-1}(s) \tag{10}$$

and is called invariant equivalent (SMITH-MCMILLAN) form of the *MIMO* process. The number of nonzero element r in the main diagonal of these matrices is the so-called

MCMILLAN-*degree*. It is obvious that $\mathcal{A}(s)$ is the least common multiple of the denominators and $n \geq r$. (The typesetting italic bold means *MIMO* TFM's, italic bold Euclid fonts mean *MIMO* polynomial matrices, whose product is generally not interchangeable !!!)

The SMITH-form (6) is a good basis to rewrite the TFM $\boldsymbol{P}(s)$ into equivalent *MFD*'s, where the parts of the representation are matrix polynomial matrices only. The TFM $\boldsymbol{P}(s)$ has two equivalent *MFD* forms

$$\boldsymbol{P}(s) = \boldsymbol{\mathcal{N}}_{\mathrm{R}}(s)\boldsymbol{\mathcal{D}}_{\mathrm{R}}^{-1}(s) = \boldsymbol{\mathcal{D}}_{\mathrm{L}}^{-1}(s)\boldsymbol{\mathcal{N}}_{\mathrm{L}}(s) \tag{11}$$

where $\boldsymbol{\mathcal{N}}_{\mathrm{R}}(s)$ and $\boldsymbol{\mathcal{D}}_{\mathrm{R}}(s)$ are the right, $\boldsymbol{\mathcal{N}}_{\mathrm{L}}(s)$ and $\boldsymbol{\mathcal{D}}_{\mathrm{L}}(s)$ are the left *MFD* polynomials, which can be – at least formally – easily computed

$$\boldsymbol{\mathcal{D}}_{\mathrm{L}}(s) = \boldsymbol{\mathcal{D}}_{\mathrm{D}}(s)\boldsymbol{\mathcal{U}}^{-1}(s) \quad ; \quad \boldsymbol{\mathcal{N}}_{\mathrm{L}}(s) = \boldsymbol{\mathcal{D}}_{\mathrm{N}}(s)\boldsymbol{\mathcal{V}}(s) \tag{12}$$

and

$$\boldsymbol{\mathcal{D}}_{\mathrm{R}}(s) = \boldsymbol{\mathcal{V}}^{-1}(s)\boldsymbol{\mathcal{D}}_{\mathrm{D}}(s) \quad ; \quad \boldsymbol{\mathcal{N}}_{\mathrm{R}}(s) = \boldsymbol{\mathcal{U}}(s)\boldsymbol{\mathcal{D}}_{\mathrm{N}}(s) \tag{13}$$

Note that the so-called transmission zeros of $\boldsymbol{P}(s)$ are the roots of $\prod_{i=1}^{r} \varepsilon_i(s)$.

In practical controller design it is reasonable to assume that the multivariable process $\boldsymbol{P}$ is factorable to an inverse stable $\boldsymbol{P}_+$ (*IS*) and to an inverse unstable $\boldsymbol{P}_-$ (*IU*) matrix operator

$$\boldsymbol{P} = \boldsymbol{P}_-\boldsymbol{P}_+ \neq \boldsymbol{P}_+\boldsymbol{P}_- \tag{14}$$

One must know that the *MIMO* plant form (14) can be rewrite into the equivalent

$$\boldsymbol{P} = \overline{\boldsymbol{P}}_+\overline{\boldsymbol{P}}_- \neq \overline{\boldsymbol{P}}_-\overline{\boldsymbol{P}}_+ \tag{15}$$

form. Based on the left and right *MFD* forms of the *MIMO* process the (14) and (15) representations can be written as

$$\boldsymbol{P}(s) = \boldsymbol{\mathcal{N}}_{\mathrm{R}}^{-}(s)\boldsymbol{\mathcal{N}}_{\mathrm{R}}^{+}(s)\boldsymbol{\mathcal{D}}_{\mathrm{R}}^{-1}(s) = \boldsymbol{\mathcal{D}}_{\mathrm{L}}^{-1}(s)\boldsymbol{\mathcal{N}}_{\mathrm{L}}^{+}(s)\boldsymbol{\mathcal{N}}_{\mathrm{L}}^{-}(s) \tag{16}$$

i.e., the factorable parts are

$$\boldsymbol{P}_-(s) = \boldsymbol{\mathcal{N}}_{\mathrm{R}}^{-}(s) \;\; ; \;\; \boldsymbol{P}_+(s) = \boldsymbol{\mathcal{N}}_{\mathrm{R}}^{+}(s)\boldsymbol{\mathcal{D}}_{\mathrm{R}}^{-1}(s) \;\; ; \;\; \overline{\boldsymbol{P}}_-(s) = \boldsymbol{\mathcal{N}}_{\mathrm{L}}^{-}(s) \;\; ; \;\; \overline{\boldsymbol{P}}_+(s) = \boldsymbol{\mathcal{D}}_{\mathrm{L}}^{-1}(s)\boldsymbol{\mathcal{N}}_{\mathrm{L}}^{+}(s) \tag{17}$$

Here we assumed that the above factorization can always be expressed in the "numerator" matrix polynomials, i.e.

$$\boldsymbol{\mathcal{N}}_{\mathrm{R}}(s) = \boldsymbol{\mathcal{N}}_{\mathrm{R}}^{-}(s)\boldsymbol{\mathcal{N}}_{\mathrm{R}}^{+}(s) \quad \text{and} \quad \boldsymbol{\mathcal{N}}_{\mathrm{L}}(s) = \boldsymbol{\mathcal{N}}_{\mathrm{L}}^{+}(s)\boldsymbol{\mathcal{N}}_{\mathrm{L}}^{-}(s) \tag{18}$$

2. ***Youla-Parametrization*** of ***MIMO*** Closed-Loop Systems

The formal generalization of the *Y-parametrization* (*YP*) [5] for linear dynamic *MIMO* control systems is relatively simple, introducing the TFM

$$\boldsymbol{Q}=\boldsymbol{C}\left(\mathbf{I}-\boldsymbol{PC}\right)^{-1}=\left(\mathbf{I}-\boldsymbol{CP}\right)^{-1}\boldsymbol{C} \tag{19}$$

and using the matrix regulator

$$\boldsymbol{C}=\left(\mathbf{I}-\boldsymbol{QP}\right)^{-1}\boldsymbol{Q}=\boldsymbol{Q}\left(\mathbf{I}-\boldsymbol{PQ}\right)^{-1} \tag{20}$$

Here $\boldsymbol{P}$ is a stable *MIMO* dynamic process (see Figure 1.), furthermore the *Y-parameter* $\boldsymbol{Q}$ and regulator $\boldsymbol{C}$ are also matrix operators (TFM's). It is not very difficult to prove that the left and right side formulation of $\boldsymbol{Q}$ and $\boldsymbol{C}$ are equivalent.

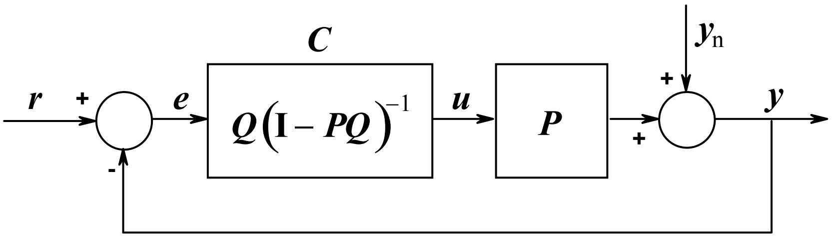

Figure 1. Generalization of the Y-parametrization

The next two relationships play important role in the closed-loop characteristics of the systems:

$$\left(\mathbf{I}+\boldsymbol{PC}\right)^{-1}=\left[\mathbf{I}+\boldsymbol{PQ}\left(\mathbf{I}-\boldsymbol{PQ}\right)^{-1}\right]^{-1}=\mathbf{I}-\boldsymbol{PQ} \quad ;$$

$$\left(\mathbf{I}+\boldsymbol{CP}\right)^{-1}=\left[\mathbf{I}+\left(\mathbf{I}-\boldsymbol{QP}\right)^{-1}\boldsymbol{QP}\right]^{-1}=\mathbf{I}-\boldsymbol{QP} \tag{21}$$

Here the following matrix identities were used:

$$\left[\mathbf{I}+\left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{A}\right]^{-1}=\mathbf{I}-\mathbf{A} \quad ; \quad \left[\mathbf{I}+\mathbf{B}\left(\mathbf{I}-\mathbf{B}\right)^{-1}\right]^{-1}=\mathbf{I}-\mathbf{B} \tag{22}$$

After some straightforward (do not miss the interchangeability in matrix operator products) manipulations

$$\boldsymbol{y}=\boldsymbol{P}\,\boldsymbol{Q}\boldsymbol{r}-\left(\mathbf{I}-\boldsymbol{PQ}\right)\boldsymbol{y}_{\mathrm{n}} \tag{23}$$

is obtained for the *YP* closed-loop system in Figure 1. Constructing a special two-degree-of-freedom (*2DOF*) scheme, shown in Figure 2 the closed-loop is "virtually" opened as

$$\boldsymbol{y} = \boldsymbol{P}\,\boldsymbol{r} - \left(\mathbf{I} - \boldsymbol{PQ}\right)\boldsymbol{y}_{\mathrm{n}} \tag{24}$$

This special scheme is called the *KB-parametrization* of a *2DOF* system. This parametrization applies internally a $\boldsymbol{Q}^{-1}$ prefilter to the *YP* scheme in Figure 1. The presented scheme and principle completely corresponds to the methods developed for *SISO* systems [1], [2]. Note that the *KB-parametrization* works for any controller $\boldsymbol{C}$, not only for YP, and provides $\boldsymbol{P}\,\boldsymbol{r}$ for the tracking properties of the *2DOF* system. The disturbance rejection properties $\left(\mathbf{I} - \boldsymbol{PQ}\right)$ can be obtained only for *YP* loop.

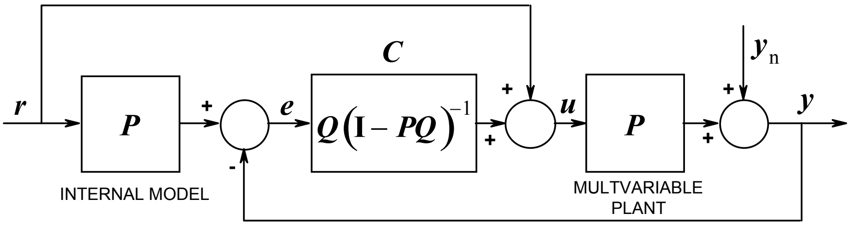

Figure 2. The general form of the K-B-parametrized MIMO 2DOF control system

3. A Generic Controller Scheme for Multivariable Processes

The general form of the *K-B-parametrized MIMO 2DOF* control system makes possible to construct a simple and effective controller scheme, which is called *generic 2DOF* (*G2DOF*) control and shown in Figure 3. The *K-B parametrization* for closed-loop control is not so widely known as the *Youla-Kucera-* (*Y-K*) *parametrization*, however, it is much closer to a control engineering view and its most important advantage in *2DF* systems is that it virtually opens the closed-loop [3], [4]. However, this parametrization can only be applied for open-loop stable processes. The introduced scheme can be referred as a *MIMO G2DF* control system.

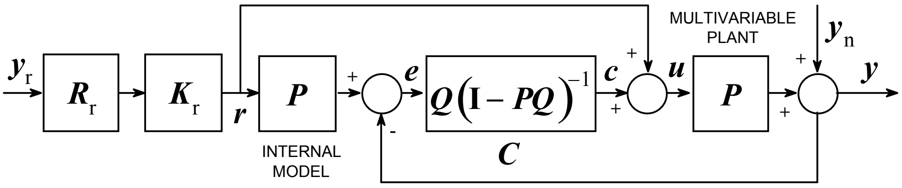

Figure 3. Generalization of the generic 2DOF control system for MIMO processes

It is interesting to see how the transfer characteristics of this *G2DOF MIMO* control system look like:

$$\boldsymbol{y} = \boldsymbol{PQ}_{\mathrm{r}}\,\boldsymbol{y}_{\mathrm{r}} - \left(\boldsymbol{I} - \boldsymbol{PQ}_{\mathrm{n}}\right)\boldsymbol{y}_{\mathrm{n}} = \boldsymbol{PK}_{\mathrm{r}}\boldsymbol{R}_{\mathrm{r}}\,\boldsymbol{y}_{\mathrm{r}} - \left(\boldsymbol{I} - \boldsymbol{PK}_{\mathrm{n}}\boldsymbol{R}_{\mathrm{n}}\right)\boldsymbol{y}_{\mathrm{n}} = \boldsymbol{y}_{\mathrm{t}} + \boldsymbol{y}_{\mathrm{d}} \tag{25}$$

If the multivariable process $\boldsymbol{P}$ is factorable according to (14) then using the proper selection of

$$\boldsymbol{Q} = \boldsymbol{Q}_{\mathrm{n}} = \boldsymbol{K}_{\mathrm{n}}\boldsymbol{R}_{\mathrm{n}} = \boldsymbol{P}_{+}^{-1}\boldsymbol{W}_{\mathrm{n}}\boldsymbol{R}_{\mathrm{n}} \tag{26}$$

and

$$\boldsymbol{Q}_{\mathrm{r}} = \boldsymbol{K}_{\mathrm{r}}\boldsymbol{R}_{\mathrm{r}} = \boldsymbol{P}_{+}^{-1}\boldsymbol{W}_{\mathrm{r}}\boldsymbol{R}_{\mathrm{r}} \quad ; \quad \boldsymbol{K}_{\mathrm{n}} = \boldsymbol{P}_{+}^{-1}\boldsymbol{W}_{\mathrm{n}} \quad ; \quad \boldsymbol{K}_{\mathrm{r}} = \boldsymbol{P}_{+}^{-1}\boldsymbol{W}_{\mathrm{r}} \tag{27}$$

the following *MIMO* regulator is obtained

$$\boldsymbol{C} = \boldsymbol{Q}\left(\mathbf{I} - \boldsymbol{P}\boldsymbol{Q}\right)^{-1} = \boldsymbol{K}_{\mathrm{n}}\boldsymbol{R}_{\mathrm{n}}\left(\mathbf{I} - \boldsymbol{P}\boldsymbol{K}_{\mathrm{n}}\boldsymbol{R}_{\mathrm{n}}\right)^{-1} = \boldsymbol{P}_{+}^{-1}\boldsymbol{W}_{\mathrm{n}}\boldsymbol{R}_{\mathrm{n}}\left(\mathbf{I} - \boldsymbol{P}_{-}\boldsymbol{W}_{\mathrm{n}}\boldsymbol{R}_{\mathrm{n}}\right)^{-1} \tag{28}$$

Based on the previous formulas the transfer characteristics of the closed-loop is now obtained as:

$$\boldsymbol{y} = \boldsymbol{P}_{-}\boldsymbol{W}_{\mathrm{r}}\boldsymbol{R}_{\mathrm{r}}\,\boldsymbol{y}_{\mathrm{r}} - \left(\mathbf{I} - \boldsymbol{P}_{-}\boldsymbol{W}_{\mathrm{n}}\boldsymbol{R}_{\mathrm{n}}\right)\boldsymbol{y}_{\mathrm{n}} = \boldsymbol{y}_{\mathrm{t}} + \boldsymbol{y}_{\mathrm{d}} \tag{29}$$

where $\boldsymbol{y}_{\mathrm{t}}$ means tracking, $\boldsymbol{y}_{\mathrm{d}}$ mean the regulating properties of the *2DOF* closed-loop system. Here $\boldsymbol{K}_{\mathrm{r}}$ and $\boldsymbol{K}_{\mathrm{n}}$ contain the $\boldsymbol{P}_{+}^{-1}$ inverse of $\boldsymbol{P}_{+}$, i.e., the invertible part of $\boldsymbol{P}$, and apply $\boldsymbol{W}_{\mathrm{r}}$ and $\boldsymbol{W}_{\mathrm{n}}$ to attenuate the influence of the invariant *MIMO* process factor $\boldsymbol{P}_{-}$ [2]. (Note that the inverses mean functional inverses and they are not simple fractions.) The above method is a straightforward generalization of the *G2DOF* control system developed for *SISO* processes [4].

If the multivariable process $\boldsymbol{P}$ is factorable according to (15) then an equivalent regulator

$$\bar{\boldsymbol{C}} = \left(\mathbf{I} - \bar{\boldsymbol{Q}}\bar{\boldsymbol{P}}\right)^{-1}\bar{\boldsymbol{Q}} = \left(\mathbf{I} - \boldsymbol{R}_{\mathrm{n}}\bar{\boldsymbol{K}}_{\mathrm{n}}\bar{\boldsymbol{P}}\right)^{-1}\boldsymbol{R}_{\mathrm{n}}\bar{\boldsymbol{K}}_{\mathrm{n}} = \left(\mathbf{I} - \boldsymbol{R}_{\mathrm{n}}\bar{\boldsymbol{W}}_{\mathrm{n}}\bar{\boldsymbol{P}}_{-}\right)^{-1}\boldsymbol{R}_{\mathrm{n}}\bar{\boldsymbol{W}}_{\mathrm{n}}\bar{\boldsymbol{P}}_{+}^{-1} \tag{30}$$

can also be derived, where

$$\bar{\boldsymbol{K}}_{\mathrm{n}} = \bar{\boldsymbol{W}}_{\mathrm{n}}\,\bar{\boldsymbol{P}}_{+}^{-1} \tag{31}$$

The transfer characteristics of the closed-loop with controller $\bar{\boldsymbol{C}}$ is the following

$$\boldsymbol{y} = \boldsymbol{P}_{-}\boldsymbol{W}_{\mathrm{r}}\boldsymbol{R}_{\mathrm{r}}\,\boldsymbol{y}_{\mathrm{r}} - \left(\mathbf{I} - \boldsymbol{R}_{\mathrm{n}}\bar{\boldsymbol{W}}_{\mathrm{n}}\bar{\boldsymbol{P}}_{-}\right)\boldsymbol{y}_{\mathrm{n}} = \boldsymbol{y}_{\mathrm{t}} + \bar{\boldsymbol{y}}_{\mathrm{d}} \tag{32}$$

It is easy to see that the $\boldsymbol{y}_{\mathrm{t}}$ is unchanged (because $\boldsymbol{K}_{\mathrm{r}}$ is unchanged), however $\bar{\boldsymbol{y}}_{\mathrm{d}}$ is different.

In the sequel two useful applications, which are connected to further simplified *MIMO* models, will be presented.

4. A *G2DOF* Controller Using the "Naive" *MIMO* Process Model

The *MIMO* process model (2) is the simplest (some times called *naive*) *MIMO* model, which is not irreducible, however does not involve the more complex forms of the left and right *MFD*'s. Its factorable form corresponding to (21) is

$$\boldsymbol{P}(s) = \boldsymbol{P}_{-}\boldsymbol{P}_{+} = \frac{1}{\mathcal{A}(s)}\boldsymbol{\mathcal{B}}(s) = \frac{1}{\mathcal{A}(s)}\boldsymbol{\mathcal{B}}_{-}(s)\boldsymbol{\mathcal{B}}_{+}(s) \tag{33}$$

If we assume that the plant is the simplest "invertible" then the model is given by

$$\boldsymbol{P} = \frac{1}{\mathcal{A}(s)}\boldsymbol{\mathcal{B}}_{+}(s) \quad ; \quad \boldsymbol{\mathcal{B}}_{+} = \boldsymbol{\mathcal{B}} \quad ; \quad \boldsymbol{\mathcal{B}}_{-} = \mathbf{I} \tag{34}$$

Introduce the following reference model transfer matrices

$$\boldsymbol{R}_{\mathrm{r}} = \frac{1}{\mathcal{A}_{\mathrm{r}}(s)}\boldsymbol{\mathcal{B}}_{\mathrm{r}}(s) \quad \text{and} \quad \boldsymbol{R}_{\mathrm{n}} = \frac{1}{\mathcal{A}_{\mathrm{n}}(s)}\boldsymbol{\mathcal{B}}_{\mathrm{n}}(s) \tag{35}$$

If $\boldsymbol{\mathcal{B}}_{-} = \mathbf{I}$ and $\boldsymbol{\mathcal{B}}_{+} = \boldsymbol{\mathcal{B}}$, in (33) then there is no need for further optimization [2], [4], therefore it is reasonable to use the selections $\boldsymbol{W}_{\mathrm{r}} = \boldsymbol{W}_{\mathrm{n}} = \mathbf{I}$, similarly to *SISO* systems. Now using the *Youla-parameter* matrix (22)

$$\boldsymbol{Q} = \boldsymbol{Q}_{\mathrm{n}} = \mathcal{A}(s)\boldsymbol{\mathcal{B}}^{-1}(s)\boldsymbol{R}_{\mathrm{n}} = \frac{\mathcal{A}(s)}{\mathcal{A}_{\mathrm{n}}(s)}\boldsymbol{\mathcal{B}}^{-1}(s)\boldsymbol{\mathcal{B}}_{\mathrm{n}}(s) \tag{36}$$

the optimal *YP* regulator is obtained after some straightforward manipulations as

$$\boldsymbol{C}(s) = \mathrm{A}(s)\mathbf{B}^{-1}(s)\boldsymbol{R}_{\mathrm{n}}(s)\left[\mathbf{I} - \boldsymbol{R}_{\mathrm{n}}(s)\right]^{-1} = \mathrm{A}(s)\mathbf{B}_{+}^{-1}(s)\mathbf{B}_{\mathrm{n}}(s)\left[\mathrm{A}_{\mathrm{n}}(s)\mathbf{I} - \mathbf{B}_{\mathrm{n}}(s)\right]^{-1} \tag{37}$$

For discrete-time system the naive plant model is

$$\boldsymbol{G}(z) = \frac{1}{\mathrm{A}(z)}\mathbf{B}(z) \quad ; \quad \mathbf{B}_{+} = \mathbf{B} \quad ; \quad \mathbf{B}_{-} = z^{-d}\mathbf{I} \tag{38}$$

which results in the following matrix regulator

$$\begin{aligned} \boldsymbol{C}(z) &= \mathcal{A}(z)\boldsymbol{\mathcal{B}}^{-1}(z)\boldsymbol{\mathcal{B}}_{\mathrm{n}}(z)\left[\square_{\mathrm{n}}(z)\mathbf{I} - z^{-d}\boldsymbol{\mathcal{B}}_{\mathrm{n}}(z)\right]^{-1} = \\ &= \mathcal{A}(z)\boldsymbol{\mathcal{B}}_{+}^{-1}(z)\boldsymbol{\mathcal{B}}_{\mathrm{n}}(z)\left[\mathcal{A}_{\mathrm{n}}(z)\mathbf{I} - z^{-d}\boldsymbol{\mathcal{B}}_{\mathrm{n}}(z)\right]^{-1} \end{aligned} \tag{39}$$

It is interesting to note that the similarly computed *SISO* discrete-time controller is

$$\begin{aligned} C &= A(z)B^{-1}(z)B_{\mathrm{n}}(z)\left[A_{\mathrm{n}}(z) - z^{-d}B_{\mathrm{n}}(z)\right]^{-1} = \\ &= A(z)B_{+}^{-1}(z)B_{\mathrm{n}}(z)\left[A_{\mathrm{n}}(z) - z^{-d}B_{\mathrm{n}}(z)\right]^{-1} \end{aligned} \tag{40}$$

5. A *MIMO* Process Model Linear in Parameter Matrices

It is an important observation that the left *MFD* form of a discrete-time pulse TFM

$$\boldsymbol{G}(z) = \boldsymbol{\mathcal{D}}_{\mathrm{L}}^{-1}(z)\boldsymbol{\mathcal{N}}_{\mathrm{L}}(z) \tag{41}$$

can be easily used for constructing a vector difference equation by computing the left *MFD* of backward shift operator z^{-1} and normalizing by the leading matrix coefficient $\mathbf{D}_{\mathrm{L}}^{0}$, i.e.

$$\begin{gathered}\mathcal{D}_{\mathrm{L}}\left(z^{-1}\right)=\mathbf{I}+\mathbf{D}_{\mathrm{L}}^{1}z^{-1}+\mathbf{D}_{\mathrm{L}}^{2}z^{-2}+\ldots+\mathbf{D}_{\mathrm{L}}^{m}z^{-m}=\mathbf{I}+\tilde{\mathcal{D}}_{\mathrm{L}}\left(z^{-1}\right)\\ \mathcal{N}_{\mathrm{L}}\left(z^{-1}\right)=\mathbf{N}_{\mathrm{L}}^{1}z^{-1}+\mathbf{N}_{\mathrm{L}}^{2}z^{-2}+\ldots+\mathbf{N}_{\mathrm{L}}^{m}z^{-m}\end{gathered}\tag{42}$$

The m-order vector difference equation using (42)

$$\begin{gathered}\boldsymbol{y}\left[k\right]=\mathcal{N}_{\mathrm{L}}\left(z^{-1}\right)\boldsymbol{u}\left[k\right]-\tilde{\mathcal{D}}_{\mathrm{L}}\left(z^{-1}\right)\boldsymbol{y}\left[k\right]=\mathbf{N}_{\mathrm{L}}^{1}\,\boldsymbol{u}\left[k-1\right]+\mathbf{N}_{\mathrm{L}}^{2}\,\boldsymbol{u}\left[k-2\right]+\ldots+\mathbf{N}_{\mathrm{L}}^{m}\,\boldsymbol{u}\left[k-m\right]-\\ -\mathbf{D}_{\mathrm{L}}^{1}\,\boldsymbol{y}\left[k-1\right]+\mathbf{D}_{\mathrm{L}}^{2}\,\boldsymbol{y}\left[k-2\right]+\ldots+\mathbf{D}_{\mathrm{L}}^{m}\,\boldsymbol{y}\left[k-m\right]\end{gathered}\tag{43}$$

is linear in the matrix parameters $\mathbf{N}_{\mathrm{L}}^{i}$ and $\mathbf{D}_{\mathrm{L}}^{i}$, therefore it is easy to construct a *Least-Squares* parameter estimation method to identify the *MIMO* process model [3]. The *MIMO* controller (27) can be used to form a two-step procedure where the output $\boldsymbol{c}\left[k\right]$ of the controller can be computed via an auxiliary variable $\boldsymbol{x}\left[k\right]$.

Assuming the simpler naive process model (34) and a left *MFD* reference model $\boldsymbol{R}_{\mathrm{n}}$ as

$$\boldsymbol{R}_{\mathrm{n}}=\mathcal{A}_{\mathrm{n}}^{-1}(z)\mathcal{B}_{\mathrm{n}}(z)=\left[\mathbf{I}+\tilde{\mathcal{A}}_{\mathrm{n}}\left(z^{-1}\right)\right]^{-1}\mathcal{B}_{\mathrm{n}}\left(z^{-1}\right)\tag{44}$$

the basic controller equations can be obtained in the forms of

$$\boldsymbol{c}=\bar{\boldsymbol{C}}\,\boldsymbol{e}=\left(\mathbf{I}-\boldsymbol{R}_{\mathrm{n}}\bar{\boldsymbol{P}}_{-}\right)^{-1}\boldsymbol{R}_{\mathrm{n}}\bar{\boldsymbol{P}}_{+}^{-1}\boldsymbol{x}\quad;\quad\boldsymbol{x}=\bar{\boldsymbol{P}}_{+}^{-1}\boldsymbol{e}\tag{45}$$

These equations are easy to rewrite into the vector difference equations

$$\boldsymbol{c}=\left(\mathcal{B}_{\mathrm{n}}\mathcal{N}_{\mathrm{L}}^{-}-\tilde{\mathcal{A}}_{\mathrm{n}}\right)\boldsymbol{c}+\mathcal{B}_{\mathrm{n}}\,\boldsymbol{x}\tag{46}$$

and

$$\boldsymbol{x}=\mathcal{D}_{\mathrm{L}}\,\boldsymbol{e}-\tilde{\mathcal{N}}_{\mathrm{L}}^{+}\,\boldsymbol{x}\tag{47}$$

Here the following equations were used

$$\bar{\boldsymbol{P}}_{+}^{-1}=\left[\mathcal{N}_{\mathrm{L}}^{+}(s)\right]^{-1}\mathcal{D}_{\mathrm{L}}(s)\quad;\quad\bar{\boldsymbol{P}}_{-}=\mathcal{N}_{\mathrm{L}}^{-}\quad;\quad\mathcal{N}_{\mathrm{L}}^{+}=\mathbf{I}+\tilde{\mathcal{N}}_{\mathrm{L}}^{+}\tag{48}$$

6. Examples

Example 1. Let us consider a very simple example first, when the plant transfer function matrix is

$$\boldsymbol{P}(s)=\frac{1}{\mathcal{A}(s)}\boldsymbol{\mathcal{B}}(s)=\begin{bmatrix}\dfrac{1}{1+s} & \dfrac{1}{1+2s}\\ 0 & \dfrac{1}{1+4s}\end{bmatrix}=\frac{1}{(1+s)(1+2s)(1+4s)}\begin{bmatrix}(1+2s)(1+4s) & (1+s)(1+4s)\\ 0 & (1+s)(1+2s)\end{bmatrix}$$

(49)

Select a speeding up and decoupling reference model

$$\boldsymbol{R}_{\mathrm{n}}(s)=\frac{1}{\mathcal{A}_{\mathrm{n}}(s)}\boldsymbol{\mathcal{B}}_{\mathrm{n}}(s)=\frac{1}{(1+0.5s)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\frac{1}{(1+0.5s)}\mathbf{I} \tag{50}$$

The detailed computation of the transfer function matrix of the *MIMO* regulator using equation (37) gives

$$\boldsymbol{C}(s)=\begin{bmatrix}\dfrac{1+s}{0.5s} & -\dfrac{(1+s)(1+4s)}{0.5s(1+2s)}\\ 0 & \dfrac{1+4s}{0.5s}\end{bmatrix} \tag{51}$$

So the obtained regulator contains *PI* like regulators in two and a *PID* like in one element.

Example 2. Investigate a discrete-time process now. Let the matrix pulse transfer function of the plant be

$$\boldsymbol{G}(z)=\begin{bmatrix}\dfrac{0.5z^{-1}}{1-0.5z^{-1}} & \dfrac{0.2z^{-1}}{1-0.8z^{-1}}\\ 0 & \dfrac{z^{-1}-0.5z^{-2}}{1-1.7z^{-1}+0.2z^{-1}}\end{bmatrix} \tag{52}$$

and apply again a speeding up and decoupling reference model

$$\boldsymbol{R}_{\mathrm{n}}(z)=\begin{bmatrix}\dfrac{0.8z^{-1}}{1-0.2z^{-1}} & 0\\ 0 & \left(\dfrac{0.9z^{-1}}{1-0.1z^{-1}}\right)^2\end{bmatrix}=\frac{\begin{bmatrix}0.8z^{-1}\left(1-0.1z^{-1}\right)^2 & 0\\ 0 & \left(0.9z^{-1}\right)^2\left(1-0.2z^{-1}\right)\end{bmatrix}}{\left(1-0.2z^{-1}\right)\left(1-0.1z^{-1}\right)^2} \tag{53}$$

The detailed computation of the pulse transfer function matrix of the *MIMO* regulator using equation (37) gives

$$\boldsymbol{C}(z)=\begin{bmatrix}C_{11}(z) & C_{12}(z)\\ C_{21}(z) & C_{22}(z)\end{bmatrix} \tag{54}$$

where

$$\boldsymbol{C}_{11}(z)=\frac{1.6\left(1-0.5z^{-1}\right)}{1-z^{-1}} \quad ; \quad \boldsymbol{C}_{12}(z)=\frac{-0.32\left(1-1.7z^{-1}+0.2z^{-2}\right)}{\left(1-z^{-1}\right)\left(1-0.8z^{-1}\right)} \tag{55}$$

$$\boldsymbol{C}_{21}(z)=0 \quad , \quad \boldsymbol{C}_{22}(z)=\frac{0.81z^{-1}\left(1-1.7z^{-1}+0.2z^{-2}\right)}{\left(1-z^{-1}\right)\left(1+0.8z^{-1}\right)\left(1-0.5z^{-1}\right)} \tag{56}$$

So all elements of the regulator are realizable, because of the special selection of the matrix reference model elements. These sub-regulators are integrating, because the matrix reference model $\boldsymbol{R}_{\mathrm{n}}$ has unity gain nontrivial elements. (These examples are very simple low order ones, because the typesetting for higher order plants is very difficult, the regulator elements are much higher order then.)

7. Conclusions

A formal generalization of the *Youla-parametrization* for *MIMO* systems is presented. The matrix transfer function form of the *Youla-parameter* was introduced and the design of a *MIMO* regulator was also developed for the *KB-parametrization* based generic *2DOF* scheme. The optimal regulator is given for continuous and discrete-time case, too. The controller equations are further simplified for a commutative naive *MIMO* plant model.

A continuous-time and a discrete-time simple example is shown to ease the understanding of the methodology.

References

[1] Keviczky, L.: *Combined identification and control: another way*. (Invited plenary paper.) 5th IFAC Symp. on Adaptive Control and Signal Processing, ACASP'95, Budapest, H, (1995), pp. 13-30.

[2] Keviczky, L., Bányász, CS.: *Optimality of two-degree of freedom controllers in* H_2*- and* $\boldsymbol{H}_\infty$*-norm space, their robustness and minimal sensitivity*. 14th IFAC World Congress, **F**, Beijing, PRC, (1999), pp. 331-336.

[3] Keviczky, L., Bányász, CS.: *Closed-loop parametrization schemes: Identification of the K-B and d-Y parameters*. 3rd IFAC Symposium on Robust Control Design, ROCOND'2000, CD printing, Prague, CZ, (2000).

[4] Keviczky, L., Bányász, CS.: *On the* H_2, $\boldsymbol{L}_2$ *and* $\boldsymbol{H}_\infty$, $\boldsymbol{L}_\infty$ *optimality of some two-degree of freedom control systems*. Journal of Systems Science, 33(3), (2008), pp. 39-49.

[5] Maciejowski, J. M.: *Multivariable Feedback Design*, Addison Wesley, (1989).

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