The Dynamic Model of the Slider-crank Mechanism

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Abstract: The paper presents a dynamic model of the slider-crank mechanism for combustion engines. The formulation is expressed by only one independent variable. The mass of the connection rod has been concentrated on two ends (two eyes). The model uses the Euler-Lagrange equation and it has been computed for numerical simulation by using MAPLE system.

Keywords: *Euler-Lagrange equations, slider-crank mechanism, connection rod, adiabatic process, turning moment*

1. Introduction

A slider-crank mechanism is used in combustion engines, it has been studied extensively in the past three decades.

Lot of methods can be found in the literature about this [1-3,5]. Geometrical approach, Newton-Euler Law, Wittenbauer method, Hamilton principle, and Lagrange multiplier are typically used.

This study is to demonstrate that the Lagrange equation by using only one independend variable (the rotation angle) can give a good and simple interpretation for the dynamic of the slider-crank mechanism. The model has been computed by using MAPLE system [7]. This basic model is the first step to develop a cylinder pressure model of the combustion engine for diagnostics (fault detection) Cold-Test in an engine plant.

By using only one independent variable the model should be running easily in real time. That is important for any application of control or diagnostics.

Table 1. Nomenclature

m_{hr}	*(kg)*	the concentrated mass in the big eye of the connecting rodmass
m_{ha}	*(kg)*	the concentrated mass in the small eye of the connecting rodmass
m_d	*(kg)*	the mass of the piston and piston pin
m_h	*(kg)*	the mass of the connection rod
l_h	*(m)*	the distance of the center of gravity with the center of the big eye of the connecting rod
m_k	*(kg)*	the mass of the crankshaft
m_r	*(kg)*	sum of rotationed mass for one cylinder
m_a	*(kg)*	sum of alterned mass for one cylinder
J_{f1}	(kgm^2)	inertia of the crank and crankshaft for 1. cyl.
J_r	(kgm^2)	inertia of the rotational masses for 1. cyl.
p_d	(N/m^2)	the gaspressure acting on the piston
D	*(m)*	diameter of the piston
F_d	*(N)*	the gas force acting on the piston
F_T	*(N)*	the tangential force
M_k	*(Nm)*	the driving moment
r	*(m)*	crank radius
l	*(m)*	the length of the rod
λ		ratio of crank to connection rod length
ε		compression ration
φ	*(rad)*	the angle position of the crankshaft
β	*(rad)*	the angle of connection rod with X-axis
A_d	(N/m^2)	the cross –sectional area of the piston
p_d	*(Pa)*	the gas pressure in the cylinder
p_0	*(Pa)*	atmospheric pressure
V_0	(m^3)	the volume of the cylinder
κ		expansion coefficient
x_d	*(m)*	displacement of the piston

2. The dynamic formulation of slider-crank mechanism

The simple model of the slider-crank mechanism (Fig. 1) consists of three parts: a crank-shaft, a connection rod, and a piston. In this study, the simple dynamic formulation is expressed by only one independent variable, of the rotation angle ϕ.

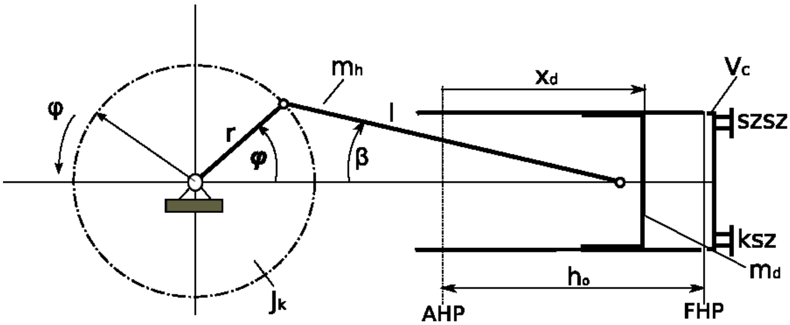

Figure 1. Slider-crank mechanism

2.1 Kinematic equations

The mechanism (Fig. 1) has a constrained condition as follows

$$r \sin\varphi = l \sin\beta \tag{1}$$

The piston has a linear motion in *x*-direction:

$$x_d = r(1 + \cos\varphi) + l(-1 + \cos\beta) \tag{2}$$

The ratio of crank to connection rod length:

$$\lambda = \frac{r}{l} \tag{3}$$

trought substitution *cos β* with function of *φ*:

$$\cos\beta = \sqrt{1 - \lambda^2 \sin^2\varphi} \tag{4}$$

So we can get the displacement of the piston with the following formula:

$$x_d = r(1 + \cos\varphi) + l(-1 + \sqrt{1 - \lambda^2 \sin^2\varphi}) \tag{5}$$

The speed of the piston by taking the first derivates of displacement (Eq. 5) can be written:

$$\dot{x}_d = -r \sin\varphi\, \dot{\varphi} - \frac{l\lambda^2 \sin\varphi \cos\varphi \dot{\varphi}}{\sqrt{1-\lambda^2 \sin^2\varphi}} \tag{6}$$

It is influenced by only one independent variable, the rotation angle ϕ.

2.2 The mass distribution in the crank mechanism

To avoid complicated calculations, the mass is divided to two parts in the crank mechanism:

1) the reciprocating masses,
2) the rotational masses.

The dynamics of any rigid body can be characterized by considering an equivalent system of finite number of particles. So the connection rod can be approximated by a system of two particles. This simple consideration can be taken from low to middle engine speed.

As the connecting rod has both transferring and rotating movements, we consider its masses concentrated on two ends (two eyes). So the mass in the small eye performs transferring movements and the one in the big eye (crank end) does rotational movement.

So the connection rod can be approximated by a system of two particles.

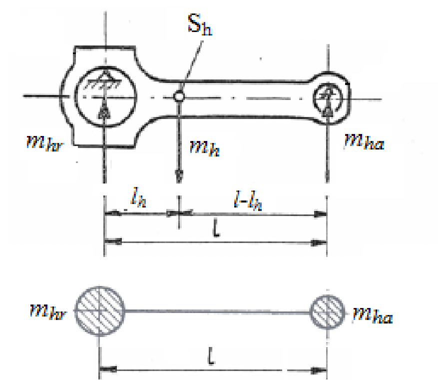

Figure 2. Approximation mass of the connection rod

If it is in the center of the mass of the connection rod Sh, as shown in the Fig. 2, the equivalent system of two particles of mass mhr and mha is given by:

$$m_{ha} = m_h \frac{l_h}{l} \quad (7)$$

$$m_{hr} = m_h \frac{l-l_h}{l} \quad (8)$$

where m_h is the original mass of the connection rod.

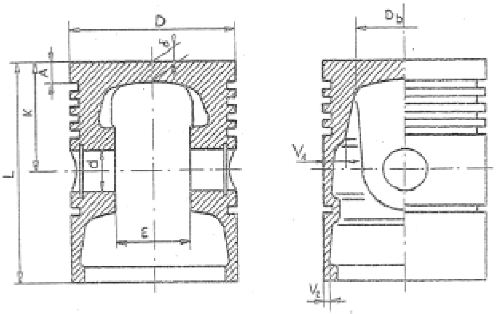

Figure 3. The piston

The reciprocating mass as sum of mass of the piston (Fig. 3.) and the concentrated mass in the small eye of the connecting rodmass is expressed with:

$$m_a = m_d + m_{ha} \tag{9}$$

The inertia of the rotational masses (Fig. 4.) can be obtained by:

$$J_r = J_{f1} + m_{hr} r^2 \tag{10}$$

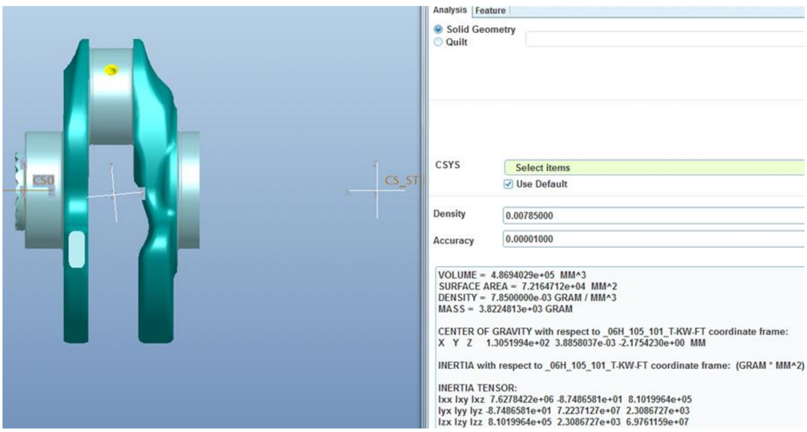

Figure 4. The inertia of the crank and crankshaft for 1. cyl. has been computed in ProEE. (from Audi 2.0L TFSI 4cyl. Engine)

2.3 The Euler–Lagrange equation

The Euler–Lagrange equation will be applied in the following form:

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{q}_i} - \frac{\partial E}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_{ni} \tag{11}$$

where:

E: the total kinetic energy of the system
U: the total potential energy of the system
Q_{ni}: the generalised constrained reaction force
W: the virtual works

The generalised constrained reaction force can be written:

$$\mathrm{Q}_{ni} = \sum \frac{\partial W}{\partial Q_i} \tag{12}$$

<u>Specification of the kinetic energy</u>

The total kinetic energy of the mechanism is given by:

$$E = \frac{1}{2} J_r \dot{\varphi}^2 + \frac{1}{2} m_a \dot{x}_d^2 \tag{13}$$

The generalised coordinate is:

$$\dot{q} = \dot{\varphi} \tag{14}$$

From (Eq. 6) trough substituting $\dot{x}$ to (Eq. 13) we can get:

$$E = \frac{1}{2} J_r \dot{\varphi}^2 + \frac{1}{2} m_a (-r \sin\varphi\, \dot{\varphi} - \frac{l\lambda^2 \sin\varphi \cos\varphi \dot{\varphi}}{\sqrt{1-\lambda^2 \sin^2\varphi}})^2 \tag{15}$$

Substituting (Eq. 15) into the Euler-Lagrange equation (Eq. 11) we have the following formula:

$$-J_r \ddot{\varphi} - m_a \left(-r\cos\varphi\, \dot{\varphi}^2 - r\sin\varphi\, \ddot{\varphi} - \frac{l\lambda^4 m^2 \varphi \cos^2\varphi \dot{\varphi}^2}{\sqrt{(1-\lambda^2 \sin^2\varphi)^3}} - \frac{l\lambda^2 \cos^2\varphi \dot{\varphi}^2}{\sqrt{1-\lambda^2 \sin^2\varphi}} + \frac{l\lambda^2 \sin^2\varphi \dot{\varphi}^2}{\sqrt{1-\lambda^2 \sin^2\varphi}} - \frac{l\lambda^2 \sin\varphi \cos\varphi \ddot{\varphi}}{\sqrt{1-\lambda^2 \sin^2\varphi}} \right) \left(-r\sin\varphi - \frac{l\lambda^2 \sin\varphi \cos\varphi}{\sqrt{1-\lambda^2 \sin^2\varphi}} \right) \tag{16}$$

<u>External forces, moments:</u>

Denote the pressure in the in-cylinder by pd. The piston moves from BDC to TDC (Fig. 5), compressing the gas in the cylinder. No heat exchange occurs between the gas and its surroundings. So the process is considered as adiabatical.

The pressure in the in-cylinder is manifested by:

$$p_d = p_0 \left(\frac{V_0}{V_x}\right)^{\kappa} = p_0 \left(\frac{h_0}{h_0 - xd}\right)^{\kappa} \tag{17}$$

The formula of the gasforce is:

$$F_d = (p_x - p_0) A_d = p_0 A_d \left[\left(\frac{h_0}{h_0 - xd}\right)^{\kappa} - 1 \right] \tag{18}$$

The tangential force can be expressed by :

$$F_T = F_d \sin\varphi \frac{1+\lambda\cos\varphi}{\sqrt{1-\lambda^2 \sin^2\varphi}} \tag{19}$$

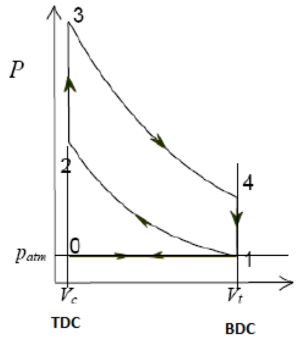

Figure 5. P-V diagram for the ideal air cycle the four stroke internal combustion engine

The gas moment acting on the crankshaft:

$$M_d = r\, F_t \tag{20}$$

Denoting the driving moment acting on the crankshaft:

$$M_k \tag{21}$$

So the generalized force is written with:

$$Q = M_d + M_K \tag{22}$$

Finally the Euler–Lagrange equation can be obtained in the following form:

$$-J_r\ddot{\varphi} - m_a\left(-r\cos\varphi\,\dot{\varphi}^2 - r\sin\varphi\,\ddot{\varphi} - \frac{l\,\lambda^4 m^2\varphi\cos^2\varphi\dot{\varphi}^2}{\sqrt{(1-\lambda^2\sin^2\varphi)^3}} - \frac{l\,m^2\cos^2\varphi\,\dot{\varphi}^2}{\sqrt{1-\lambda^2\sin^2\varphi}} + \frac{l\lambda^2\sin^2\varphi\dot{\varphi}^2}{\sqrt{1-\lambda^2\sin^2\varphi}} - \frac{l\sin\varphi\cos\varphi\,\ddot{\varphi}}{\sqrt{1-\lambda^2\sin^2\varphi}}\right)$$

$$\left(-r\sin\varphi - \frac{l\,\lambda^2\sin\varphi\cos\varphi}{\sqrt{1-\lambda^2\sin^2\varphi}}\right) = p_0 A_d\left[\left(\frac{h_0}{h_0 - xd}\right)^{\kappa} - 1\right]\sin\varphi\,\frac{1+\lambda\cos\varphi}{\sqrt{1-\lambda^2\sin^2\varphi}} \tag{23}$$

3. The numerical simulation in MAPLE

For the numerical simulation the data has been taken from the Audi 2.0L TFSI 4cyl. engine.

The Euler-Lagrange equation given by (Eq. 23), which specifies the applied forces, turning moments, and initial conditions we solve using rkf45 method built in MAPLE16.

The input data of the engine for the simulation is the following:

$$h_0 = 0{,}13\,m\,;\;\; r = 0{,}054\,m\,;\;\; l = 0{,}144\,m\,;\;\; \lambda = \frac{r}{l} = \frac{0{,}054}{0{,}144} = 0{,}0375;$$

$\varepsilon = 9{,}6\ ;\ D = 0{,}082\ m\ ;\ \ l_h = 0{,}144\ m\ ;\ \ m_d = 0{,}456\ kg\ ;$

$m_h = 0{,}568\ kg\ ;\ \ J_{f1} = 0{,}007627\ kgm^2\ ;\ \ \kappa{=}1{,}4\ ;\ p_0 = 10^5 Pa\ .$

The result of the simulation is showed on the following measured curves (6a., 6b., 7a, 7b., 8a., 8b., 9.):

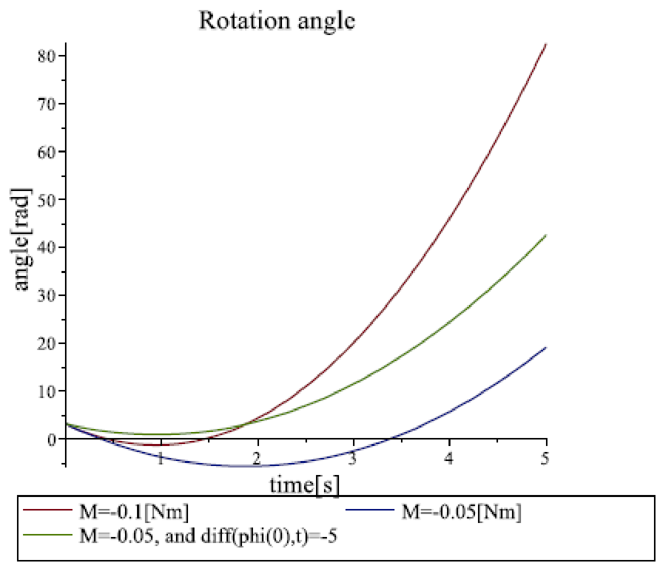

Figure 6a. Rotation angle „without gas pressure"

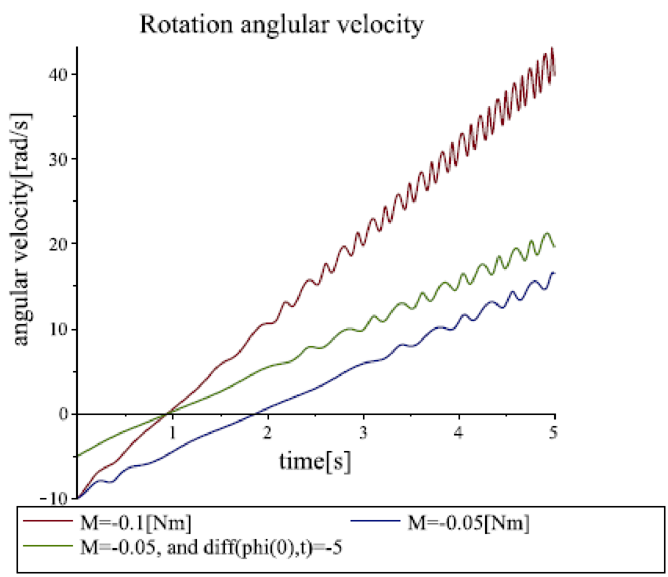

Figure 6b. Rotation angular velocity „without gas pressure"

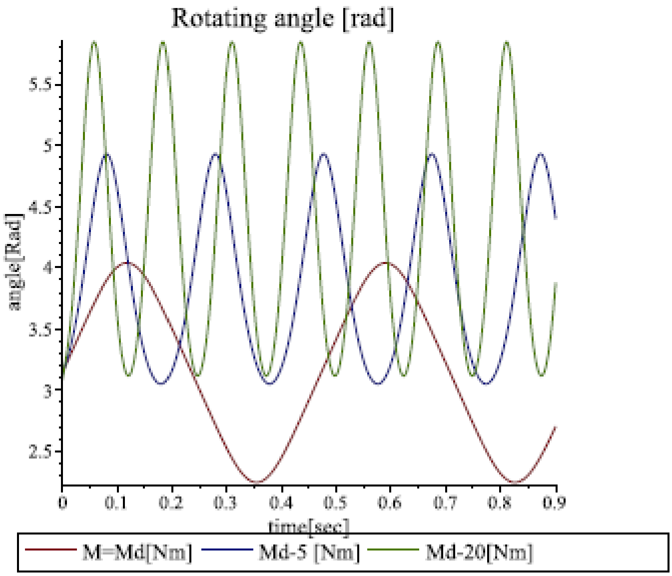

Figure 7a. Rotation angle „with gas pressure"

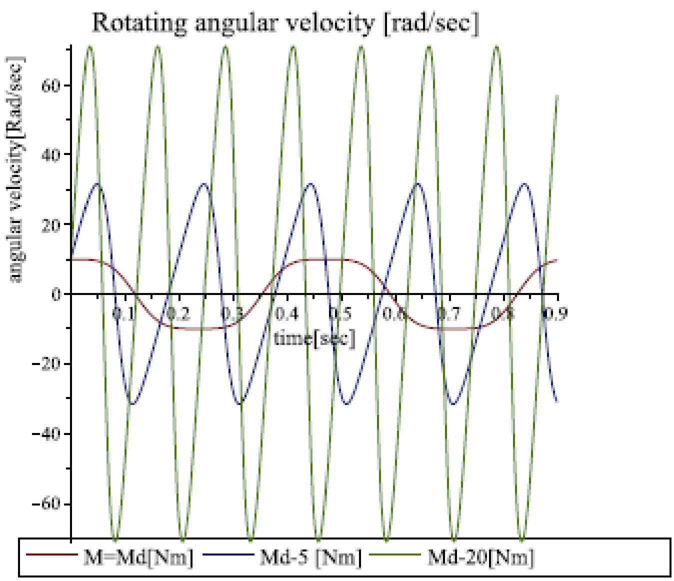

Figure 7b. Rotation angular velocity „with gas pressure"

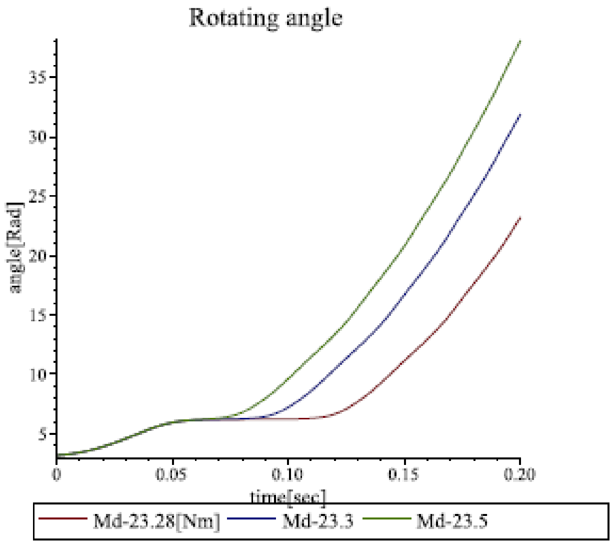

Figure 8a. Rotation angle by gas pressure by arising the driving moment

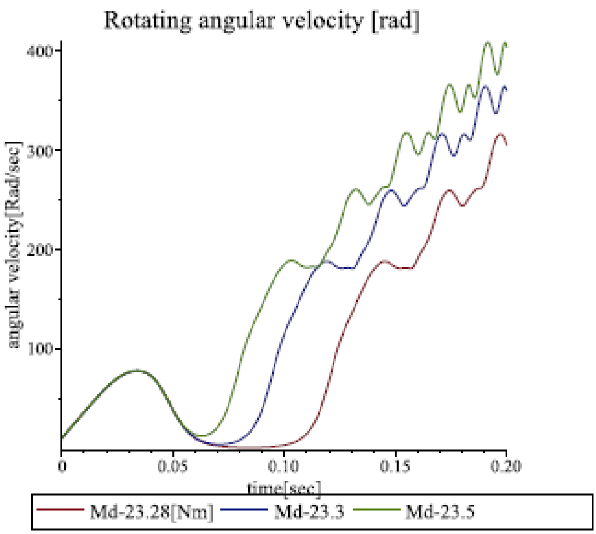

Figure 8b. Rotation angular velocity by gas pressure by arising the driving moment

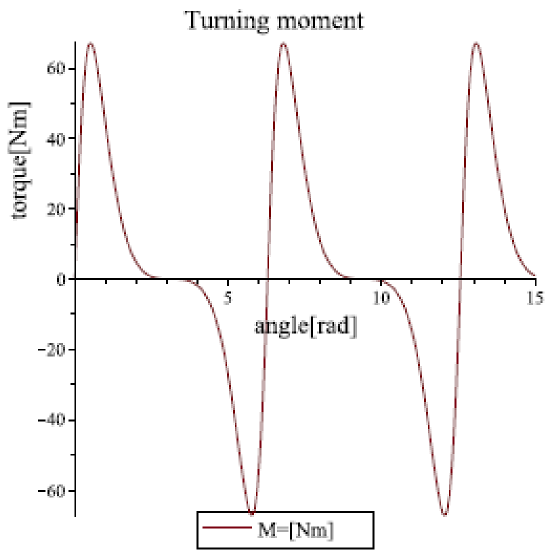

Figure 9. Turning moment

4. Results and discussion

On the Figure 6a. we can see the rotation angle depending on time φ(t), the gas pressure forces doesn't appear when the cylinder is empty. There is a little driving torque on the crankshaft, it causes the rotation of the mechanical system and accelarates it. The curves showed on the figure belong to different initial conditions.

The initial conditions φ(0)=π for each curve, then the initial conditions $\frac{d\varphi(0)}{dt}$ are different, they are

$$\frac{d\varphi(0)}{dt} = -10\,;\ \frac{d\varphi(0)}{dt} = -10\,;\ \frac{d\varphi(0)}{dt} = -5\,; \qquad (24)$$

The result of the acceleration is mirrored by the Figure 6b, where the angular velocity of the rotation can be seen for the turning torque. The oscillation of the angular velocity is the result of the mechanismus of our model.

In the case of non-empty cylinder- that is filled with gas-, the rotating torque consist of two components, the rotating torque arrising from gas compression forces and rotating torque applied on the system forces outside.

On the Figre 9. the first component of the turning moment is showed for the adiabatic gas processes. In this case the rotating angle φ(t) is showed on the Figure 7a. by the condition where the applied torque from outside is such small, that it can not cause the rotation of the system. It is clear from the calculation and the figure, that by these conditions, the system will oscillate. The frequency and its amplitude depend on the magnitude of applied torque. In our case they are M={0 ; -5 ; -20 Nm} .

A very interesting situation is, when the applied torque is greater than the maximum value of the torque arrising from gas pressure. At this situation the system begins to rotate and this rotation movement will accelerate. The acceleration is very high because of the of the large value of applied torque. On the Figures 8a. and 8b. we present this situation. For our mechanical system specificed by the data given at the beginning of this section the critical torque outside is near -23,28 [Nm]. A little larger torque applied on the system causes rotation. This is a slow process but the rotation accelerates very fast.

From these figures it can be furtherly seen, that the angular velocity oscillation becomes assymetric and can cause vibration forces.

5. Conclusion

In this work we have constructed a mechanical mathematical and a computer model of one cylinder, a slider-crank mechanism. The model has been solved with real mechanical data by using the numerical package rkf45 built in MAPLE16 software.

The results obtained are in well accordance with the results of real mechanical system. The analysis of the results shows some interesting effects such as the angular velocity oscillation and its assymetry in the presence of gas compression forces. The model and its numerical solution methodology presented here can be the starting point for construction of more complicated and more realistic models.

This model is mathematically compact enough to run in real time, and can be used as an embedded model within a control algorithm or an observer.

Acknowledgements

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References

[1] Ha, J.l., Fung, R.F., Chen, K.Y., Hsien, S.C. : *Dynamic modelling and identification of a slider-crank mechanism*, Journal of Sound and Vibration, vol. 289, pp. 1019-1044, 2006

[2] Koser, K. : *A slider-crank mechanism based robot arm performance and dynamic analysis*, Mechanism and Machine Theory, vol. 39, pp. 169-182, 2004

[3] Heywood, J.B.: *Internal combustion engine fundamentals*, McGraw-Hill, New York, 1988

[4] Doughty, S.: Mechanics of Machines, John Wiley & Sons, New York, 1988

[5] Dresig, H., Holzweig, F. : *Dynamics of Machinery*, Springer, Berlin, 2010

[6] Astashev, V.K. : *Dynamics and Control of Machines*, Springer, Berlin, 2000

[7] Fox, P.W. : *Mathematical Modeling with Maple*, Brooks Cole, Pacific Grove, 2011