FUNKCIONÁLISAN GRADIENS ANYAGÚ BIMODULUSÚ TÉGLALAP KERESZTMETSZETŰ GÖRBE RÚD HAJLÍTÁSA

BENDING OF CURVED BEAM WITH RECTANGULAR CROSS SECTION MADE OF DOUBLE MODULUS FUNCTIONALLY GRADIENT MATERIAL

Ecsedi István, Professor Emeritus, Miskolci Egyetem, Műszaki Mechanikai Intézet;
Baksa Attila, egyetemi docens, PhD, Miskolci Egyetem, Műszaki Mechanikai Intézet;
Habbachi Marwen, PhD hallgató, Miskolci Egyetem, Műszaki Mechanikai Intézet

ÖSSZEFOGLALÓ. A tanulmány tárgyát téglalap keresztmetszetű állandó görbületű funkcionálisan gradiens bimodulusú rúd tiszta hajlítási feladata alkotja. A rúd anyagának rugalmassági modulusa húzásra és nyomásra a radiális koordináta előírt függvénye. A semleges réteg pozícióját a geometriai méretek és a radiális normál feszültség-radiális fajlagos nyúlás kapcsolatát megadó függvény kettős határozza meg. A dolgozat megadja a radiális és a tangenciális normál feszültségek képleteit, a von Mises feszültség formuláját, továbbá a radiális és tangenciális elmozdulásokat, valamint a keresztmetszetek szögelfordulását. A kidolgozott analitikus módszer alkalmazását egy példa szemlélteti.

SUMMARY: The object of this study is the bending problem of curved beam with rectangular cross section made of functionally graded material with double modulus. The modulus of elasticity of the material of curved beam for tension and compression are the prescribed functions of the radial coordinate. The position of the neutral layer is determined by the geometric dimensions and the functions defining the relationship between the normal stress and normal strain. The paper gives the formulae for the radial and circumferential normal stresses, as well as the radial and circumferential displacements, as well as the angular rotations of cross sections. The application of the developed analytical method is illustrated by a numerical example.

Keywords: Saint-Venant bending, double modulus, functionally gradient material.

1 INTRODUCTION

Already de Saint-Venant had noted, that certain materials behave differently under tension and compression [1]. In the case of small deformations for tension and compression linear functions give the relationship between the normal stress and normal strains. (see Figure 1). The following constitutive law is valid for elastic materials with double Young modulus

$$\sigma = E_1\varepsilon \quad \varepsilon > 0 \quad \sigma = E_2\varepsilon \quad \varepsilon \le 0 \tag{1}$$

where σ is the normal stress, ε is the normal strain, E_1 and E_2 are the normal moduli of elasticity.

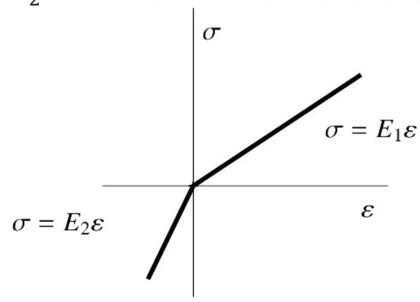

Figure 1 Stress-strain diagram.

Book of Ambartsumyan deals with the solutions for many problems of beams, disks, plates and shells [3]. Cited book of Ambarsumyan uses such constitutive law which satisfies the following symmetry conditions

$$\frac{\nu_1}{E_1} = \frac{\nu_2}{E_2}. \tag{2}$$

In equation (2) ν_1 and ν_2 are the Poisson number. Timoshenko gives the solution of bending problem for beam with straight axis [2]. The text book by Timoshenko presents closed form solution for pure bending which can be extended to nonhomogeneous bending assuming that the bending moment has constant sign, that is on the whole beam $M \ge 0$ or $M \le 0$. This paper deals with the pure bending of curved beam made of functionally grade material with double Young modulus. The material property is a second order power function of the radial coordinate r. The computations are done in the cylindrical coordinate system $Or\varphi z$, r denotes the radial coordinate, φ is the polar angle and z is the axial coordinate The unit vectors of the cylindrical coordinate system $Or\varphi z$ are $\boldsymbol{e}_r(\varphi)$, $\boldsymbol{e}_\varphi(\varphi)$ and $\boldsymbol{e}_z$ (see Figure 2).

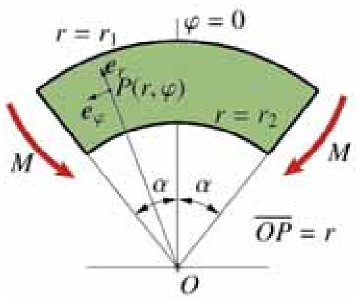

Figure 2 Pure bending of curved beam made of FG material with double modulus.

The thickness of the curved beam measured in direction of axis z is $b = const$. The Young moduli of the beam are denoted by E_1 and E_2

$$E_1 = Y_1 \left(\frac{r}{r_1}\right)^2, \qquad E_2 = Y_2 \left(\frac{r}{r_2}\right)^2, \tag{3}$$

where Y_1 and Y_2 are the Young moduli at $r = r_1$ and at $r = r_2$ respectively. On the upper part of the curved beam $\sigma_r > 0$, and on the lower part of the curved beam $\sigma_r < 0$. The position of the neutral layer is determined by the radial coordinate ρ $(r_2 \le \rho \le r_1)$.
It is evident that $\varepsilon_\varphi(\rho) = 0$ and $\sigma_\varphi(\rho) = 0$. The circumferential normal strain $\varepsilon_\varphi = \varepsilon_\varphi(r)$ can be represented as according to paper [4]

$$\varepsilon_\varphi = \frac{W}{r} + \vartheta, \quad W = \frac{\mathrm{d}^2 U}{\mathrm{d}\varphi^2} + U, \quad \vartheta = \frac{\mathrm{d}\phi}{\mathrm{d}\varphi}. \tag{4}$$

Here, $U = U(\varphi)$ is the radial displacement, $\phi = \phi(\varphi)$ is the rotation of the cross section. The possibility of the strain field given by equation (4) follows from the following displacement field

$$\boldsymbol{u}(r,\varphi,z) = U(\varphi)\boldsymbol{e}_r + (r\phi(\varphi) + V(\varphi)\boldsymbol{e}_\varphi \tag{5}$$

where $U = U(\varphi)$ is the radial displacement, $V = V(\varphi) = \frac{\mathrm{d}U}{\mathrm{d}\varphi}$ is the one of the component of circumferential displacement and $\phi = \phi(\varphi)$ is the cross sectional rotation [4]. All the strains, except for the circumferential normal strain ε_φ are zero. It is evident, in the case of pure bending ε_φ, W and ϑ do not depend on the polar coordinate φ, so that

$$\sigma_{\varphi 1}(r) = Y_1 \left(\frac{r}{r_1}\right)^2 \vartheta \left(1 - \frac{\rho}{r}\right) \quad \rho \le r \le r_1 \tag{6}$$

$$\sigma_{\varphi 2}(r) = Y_2 \left(\frac{r}{r_2}\right)^2 \vartheta \left(1 - \frac{\rho}{r}\right) \quad r_2 \le r \le \rho. \tag{7}$$

since $W = -\vartheta\rho$. The following conditions of the global equilibrium are valid

$$N = b \int_{r=\rho}^{r_1} \sigma_{\varphi 1} \mathrm{d}r + b \int_{r=r_2}^{\rho} \sigma_{\varphi 2} \mathrm{d}r = 0, \tag{8}$$

$$M = b \int_{r=\rho}^{r_1} r\sigma_{\varphi 1} \mathrm{d}r + b \int_{r=r_2}^{\rho} r\sigma_{\varphi 2} \mathrm{d}r\,. \tag{9}$$

2 SOLUTION OF THE PURE BENDING PROBLEM OF CURVED BEAM

At first, we give the detailed form of equation (8) which is

$$\begin{aligned} \int_{r=\rho}^{r_1} E_1 \left(1 - \frac{\rho}{r}\right) \mathrm{d}r + \int_{r=r_2}^{\rho} E_2 \left(1 - \frac{\rho}{r}\right) \mathrm{d}r = \\ = \int_{r=\rho}^{r_1} Y_1 \left(\frac{r}{r_1}\right)^2 \left(1 - \frac{\rho}{r}\right) \mathrm{d}r + \\ \int_{r=r_2}^{\rho} Y_2 \left(\frac{r}{r_2}\right)^2 \left(1 - \frac{\rho}{r}\right) \mathrm{d}r = 0. \end{aligned} \tag{10}$$

After some elementary computations equation (10) can be written in the form

$$\left(\frac{Y_1}{r_1^2} - \frac{Y_2}{r_2^2}\right)\rho^3 + 3(Y_2 - Y_1) + 2(Y_1 r_1 - Y_2 r_2) = 0. \tag{11}$$

The root of equation (11) which satisfies the condition $r_2 \le \rho \le r_1$ gives the position of the neutral layer. We note that ρ does not depend on the value of the applied bending moment.

Moment equilibrium equation can be written the following form

$$M = \vartheta Q, \tag{12}$$

where

$$\begin{aligned} Q = Y_1 b \left(\frac{1}{12}\frac{\rho^4}{r_1^2} + \frac{1}{4} r_1^2 - \frac{1}{3} r_1 \rho\right) \\ - Y_2 b \left(\frac{1}{12}\frac{\rho^4}{r_2^2} + \frac{1}{4} r_2^2 \\ - \frac{1}{3} r_2 \rho\right). \end{aligned} \tag{13}$$

Expression of the circumferential normal stress can be given as

$$\sigma_{\varphi 1}(r) = E_1(r)\vartheta\left(1 - \frac{\rho}{r}\right) \quad \rho \le r \le r_1, \tag{14}$$

$$\sigma_{\varphi 2}(r) = E_2(r)\vartheta\left(1 - \frac{\rho}{r}\right) \quad r_2 \le r \le \rho, \tag{15}$$

$$\begin{aligned} \sigma_\varphi(r) = (h(r - r_2) - h(r - \rho)\sigma_{\varphi 2}(r) \\ + h(r - \rho)\sigma_{\varphi 1}(r), \end{aligned} \tag{16}$$

Here, $h = h(r)$ is the Heaviside function

$$h(r) = 0 \quad r < 0, \qquad h(r) = 1 \quad r \ge 0. \tag{17}$$

To obtain the formula of radial normal stress σ_r the following equilibrium equation is used [5,6]

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\sigma_r) = \sigma_\varphi. \tag{18}$$

The solution of the ordinary differential equation for $\sigma_r = \sigma_r(r)$ is

$$\sigma_{r2}(r) = \frac{1}{r}\int_{r_2}^{r} \sigma_{\varphi 2}(\lambda)\mathrm{d}\lambda, \quad r_2 \le r \le \rho, \tag{19}$$

$$\sigma_{r1}(r) = \frac{1}{r}\int_{\rho}^{r_1} \sigma_{\varphi 1}(\lambda)\mathrm{d}\lambda + \frac{\rho}{r}\sigma_{r2}(\rho) \qquad \rho \le r \le r_1. \tag{20}$$

Expression of von Mises stresses σ_0 can be computed from equation (21) [5,6]

$$\sigma_0(r) = \sqrt{\sigma_r^2 - \sigma_r\sigma_\varphi + \sigma_\varphi^2} \quad r_2 \le r \le r_1. \tag{21}$$

The connection between the radial displacement $U = U(\varphi)$ and W is

$$W = \frac{\mathrm{d}^2 U}{\mathrm{d}\varphi^2} + U. \tag{22}$$

The circumferential displacement in terms of $U = U(\varphi)$ is

$$V(\varphi) = \frac{\mathrm{d}U}{\mathrm{d}\varphi}. \tag{23}$$

The solution of the differential equations (22) and (23) under the condition (Figure 2)

$$U(0) = 0, \quad V(0) = 0 \tag{24}$$

are

$$U(\varphi) = W(1 - \cos\varphi), \quad V(\varphi) = W\sin\varphi \qquad -\alpha \le \varphi \le \alpha \tag{25}$$

From equation (4) the expression of cross-sectional rotation can be computed (Figure 2)

$$\phi(\varphi) = \vartheta\varphi \quad -\alpha \le \varphi \le \alpha. \tag{26}$$

Here, it is assumed that $\phi(0) = 0$.

3 NUMERICAL EXAMPLE

The following data are used in the numerical example

$$Y_1 = 4 \times 10^{10}\ \text{Pa}, \quad Y_2 = 5 \times 10^{11}\ \text{Pa}$$
$$r_1 = 0.065\ \text{m}, \quad r_2 = 0.085\ \text{m}$$
$$\alpha = \frac{\pi}{6}, \quad b = 0.02\ \text{m}, \quad M = 500\ \text{Nm}.$$

Figure 3 shows the plot of the circumferential normal stress σ_φ as a function of r.

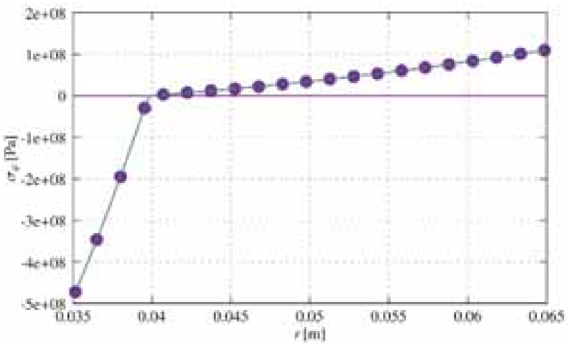

Figure 3 The graph of the circumferential normal stress.

The plot of the radial normal stress as a function of r is presented in Figure 4.

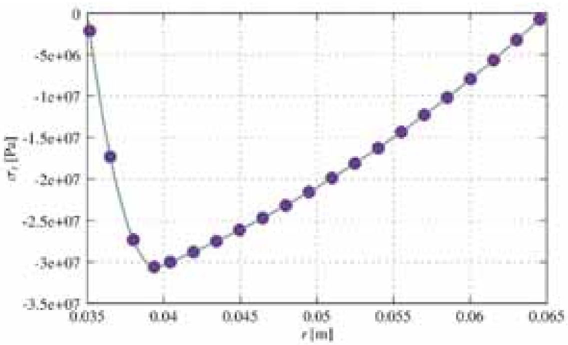

Figure 4 The plot of the radial normal stress.

The distribution of the von Mises stress is given by Figure 5.

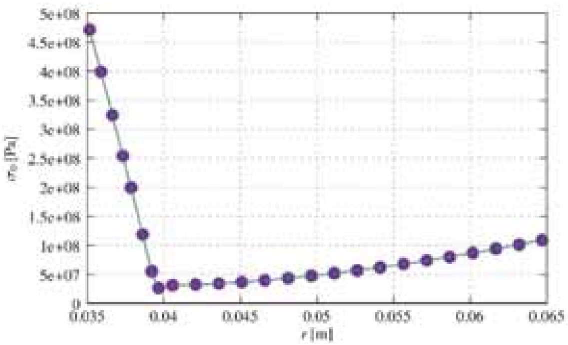

Figure 5 The plot of $\sigma_0 = \sigma_0(r)$.

The maximum value of σ_0 is $\sigma_0(r_2) = 4.7600127 \times 10^8$ Pa. The plot of the radial displacement as a function of φ is presented in Figure 6.

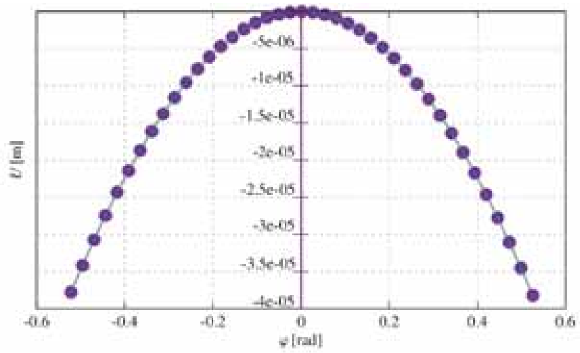

Figure 6 The graph of the $U = U(\varphi)$.

The plot of $V = V(\varphi)$ is illustrated in Figure 7.

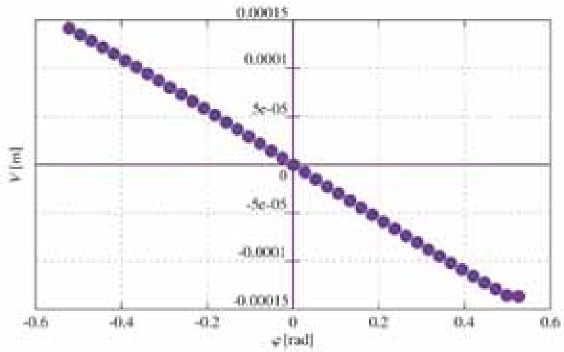

Figure 7 The plot of $V = V(\varphi)$.

The cross sectional rotation of curved beam as a function of φ is given in Figure 8.

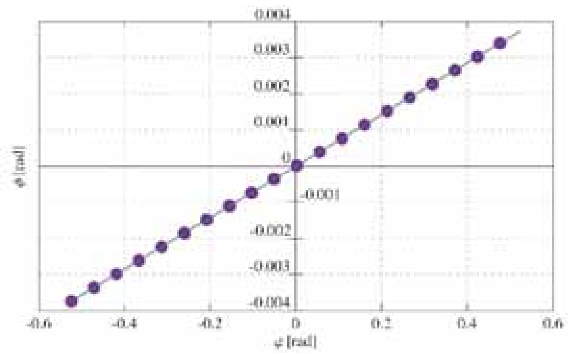

Figure 8 The plot of the cross sectional rotation $\phi = \phi(\varphi)$.

5 CONCLUSIONS

The paper deals with the pure bending of curved beam with rectangular cross section. The beam is made of functionally graded bimodulus material. The moduli of elasticity, which are different for tension and compression depend on the radial coordinate. The presented solution is obtained by the application of a strength of material formulation.

6 REFERENCES

[1] de Saint-Venant B., *Résumé Des Leçons Données À École Des Ponts Et Chaussées Sur L'application De La Mécanique Des Ma-chines*, 3rd ed., Dunod Paris, France, 1864, Vol. 1., Chapter I-IV., pp. 173-178.

[2] Timoshenko, S.P., *Strength of materials, Part II.*, 3rd ed., Mc Millan, London, UK, 1956, Chapter IX., p. 336.

[3] Ambartsumyan, S.A., *Elasticity Theory of Different Moduli.* Moscow, Nauka, Fiziko-Matematicseszkoj Literaturi, 1982.

[4] Ecsedi, I., Dluhi, K., A linear model for the static and dynamic analysis of non-homogeneous curved beams. *Applied Mathematical Modelling*, 29, (2005), pp. 1211-1231.

[5] Hosford, W., *Solid Mechanics.* Cambridge University Press, Cambridge, 2010.

[6] Rees, D.W.A., *Basic Solid Mechanics.* Macmillan Press, London, 1997.