DIMENZIÓK
Matematikai Közlemények
I. kötet, 2013

# A Diophantine Equation Including Balancing Numbers

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## 1. Introduction

The first definition of balancing numbers is essentially due to Finkelstein [5], though he named them numerical centers. A positive integer $n \geq 2$ is called balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

holds for some positive integer $r$ which is called balancer corresponding to the balancing number $n$. The $m^{th}$ term of the sequence of balancing numbers is denoted by $B_m$. It is known that the balancing numbers satisfy the recurrence

$$B_n = 6B_{n-1} - B_{n-2},$$

where the initial conditions are defined by $B_0 = 0$ and $B_1 = 1$.

The Binet formula of the balancing numbers is given by

$$B_n = \frac{\propto^n - \beta^n}{\propto - \beta},$$

where $\propto$ and $\beta$ are the roots of the characteristic polynomial $x^2 - 6x + 1$.

Behera et al. [3] showed that the diophantine equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l \tag{1}$$

has no solution in the positive integers $(n, r, k, l)$ with $n \geq 2$ in the cases $k \leq l$ and $(k, l) = (2,1), (3,1), (3,2)$. Here $F_n$ denotes $n^{th}$ Fibonacci number. They also conjectured in [3] that only the quadruples $(n, r, k, l) = (4, 3, 8, 2)$ satisfies (1). Their conjecture was proved by Alvarado et. al. [2].

We focus on the equation

$$B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l\ . \tag{2}$$

In the specific cases, we found no solutions. Based on Theorems 6-9 and an extended computer search, we conjecture the following.

**Conjecture:** There is no quadruple $(n, r, k, l)$ of positive integers $n \geq 2$ which satisfies (2).

## 2. Lemmas

The proofs of the theorems use several statements collected in this section.

**Lemma 1.** *For any positive integer* $m$

(a) $B_{m+1}B_{m-1} = (B_m + 1)(B_m - 1)$,

(b) $B_{2m-1} = B_m^2 - B_{m-1}^2$,
(c) $B_1 + B_3 + \cdots + B_{2m-1} = B_m^2$,
(d) $B_1^2 + B_2^2 + \cdots + B_{m-1}^2 < B_{m+1} + B_{m+2} + \cdots + B_{2m-3}$ for $m \geq 5$,
(e) $B_{2m+5}^2 + B_{2m+6}^2 + \cdots + B_{3m+4}^2 < B_1^3 + B_2^3 + \cdots + B_{2m+3}^3$ for $m \geq 2$,
(f) $4B_m < B_{m+1} - B_m < 5B_m$
hold.

**Lemma 2.** *Any positive integer* $n$ *satisfies*
(a) $\sum_{k=1}^{n} B_k = (B_{n+1} - B_n - 1)/4$,
(b) $\sum_{k=1}^{n} B_k^2 = (B_{2n+1} - (2n+1))/32$,
(c) $\sum_{k=1}^{n} B_k^3 = (B_{3n+3} - B_{3n} - 147(B_{n+1} - B_n) + 112)/6272$.

**Lemma 3.** *For any integer* $u \geq 3$, *the inequalities*

$$\propto^{u-0.99} < B_u < \propto^{u-0.98}$$

hold.

**Lemma 4.** *Suppose that* $a > 0$ *and* $b \geq 0$ *are real numbers and* $u_0$ *is a positive real number. Then* $\propto^u + b \leq \propto^{u+\tau}$ *holds for any* $u \geq u_0$, *where* $\tau = log_a(a + \frac{b}{\propto^{u_0}})$.

## 3. The results

Here we present four theorems and the proof of the last one. They confirm the conjecture.

**Theorem 1.** *The Diophantine equation* $B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l$ *has no solution for any positive integers* $r$ *and* $n \geq 2$ *if* $k \leq l$.

**Theorem 2.** *The Diophantine equation* $B_1^2 + B_2^2 + \cdots + B_{n-1}^2 = B_{n+1} + B_{n+2} + \cdots + B_{n+r}$ *has no solution in positive integer* $r$ *and* $n \geq 2$.

**Theorem 3.** *The Diophantine equation* $B_1^3 + B_2^3 + \cdots + B_{n-1}^3 = B_{n+1} + B_{n+2} + \cdots + B_{n+r}$ *has no solution for any positive integer* $r$ *and* $n \geq 2$.

**Theorem 4.** *The Diophantine equation* $B_1^3 + B_2^3 + \cdots + B_{n-1}^3 = B_{n+1}^2 + B_{n+2}^2 + \cdots + B_{n+r}^2$ *has no solution for any positive integer* $r$ *and* $n \geq 2$.

PROOF OF THEOREM 4. For $2 \leq n \leq 4$ the statement is obvious. Assume now $n \geq 5$. The application of Lemma 3(b) and (c) convert the equation

$$B_1^3 + B_2^3 + \cdots + B_{n-1}^3 = B_{n+1}^2 + B_{n+2}^2 + \cdots + B_{n+r}^2.$$

To

$$\frac{B_{3n} - B_{3(n-1)} - 147\left(B_n - B_{(n-1)}\right) + 112}{32 \cdot 196} = \frac{1}{32}(B_{2(n+r)+1} - B_{2n+1} - 2r).$$

Which is equivalent to

$$B_{3n} - B_{3(n-1)} - 147(B_n - B_{n+1}) + 196B_{2n+1} + 112 = 196\left(B_{2(n+r)+1} - 2r\right).$$

Put

$$LS := B_{3n} - B_{3(n-1)} - 147(B_n - B_{n+1}) + 196B_{2n+1} + 112,$$

$$RS \coloneqq 196\left(B_{2(n+r)+1} - 2r\right).$$

Now use Lemma 4 and Lemma 2 (f) to obtain

$$\propto^{2n+2r+3} < RS < \propto^{2n+2r+3.02}.$$

Similarly,

$$196B_{3(n-1)} < LS < (196+197)B_{3(n-1)},$$

$$\propto^{3n-1.01} < LS < \propto^{3n-0.59}$$

hold. By the equation above

$$\max\{2n+2r+3, 3n-1.01\} < \min\{2n+2r+3.02, 3n-0.59\}$$

follows, which yields

$$n - 4.03 < 2r < 3.59.$$

Clearly, only $2r = n - 4$ is possible. Inserting $n = 2r + 4$ to the equation, we obtain

$$B_1^3 + B_2^3 + \cdots + B_{2r+5}^3 = B_{2r+5}^2 + B_{2r+6}^2 + \cdots + B_{3r+4}^2,$$

and we arrived at a contradiction with Lemma 2(e).

## References

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