DIMENZIÓK
Matematikai Közlemények
II. kötet, 2014

Incomplete Stirling numbers

Takao Komatsu
School of Mathematics and Statistics
Wuhan University
Wuhan 430072 China
komatsu@whu.edu.cn

ÖSSZEFOGLALÓ. Az előjel nélküli elsőfajú Stirling számok általánosításával a hiányos, előjel nélküli elsőfajú Stirling számokat és az asszociált elsőfajú Stirling számokat definiáljuk. Továbbá vizsgáljuk ezen számok néhány alapvető tulajdonságát.

ABSTRACT. We define two types of incomplete unsigned Stirling numbers of the first kind, named as restricted Stirling numbers of the first kind and associated Stirling numbers of the first kind, by generalizing the (unsigned) Stirling numbers of the first kind. We also investigate some basic properties.

1. Introduction and Stirling numbers of the second kind

Denote $\left\{ n \atop k \right\}$ the Stirling numbers of the second kind ([3]), determined by

$$\left\{ n \atop k \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^i \binom{k}{j}(k-j)^n$$

([3]). In place of the classical Stirling numbers $\left\{ n \atop k \right\}$, we substitute the restricted Stirling numbers of the second kind $\left\{ n \atop k \right\}_{\le m}$ and associated Stirling numbers of the second kind $\left\{ n \atop k \right\}_{\ge m}$, respectively.

The restricted Stirling number of the second kind $\left\{ n \atop k \right\}_{\le m}$ gives the number of partitions of n elements into k subsets such that none of the blocks contain more than m elements (e.g. [5], [6]). The associated Stirling number of the second kind $\left\{ n \atop k \right\}_{\ge m}$ gives the number of partitions of an n element set into k subsets such that every block contains at least m elements ([2, p.221], [5], [7]).

Some combinatorial and congruental properties of these numbers can be found in [5], and other properties can be found in the cited papers of [5].

The generating functions of these numbers are given by

$$\sum_{n=k}^{mk} \left\{ n \atop k \right\}_{\le m} \frac{x^n}{n!} = \frac{1}{k!}\left(-1+\sum_{k=0}^{m}\frac{t^k}{k!}\right)^k$$

and

$$\sum_{n=mk}^{\infty} \left\{ {n \atop k} \right\}_{\geq m} \frac{x^n}{n!} = \frac{1}{k!}\left(e^x - \sum_{k=0}^{m-1} \frac{t^k}{k!}\right)^k,$$

respectively. Since the generating function of $\left\{ {n \atop k} \right\}$ is given by

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!},$$

we have

$$\left\{ {n \atop k} \right\}_{\leq \infty} = \left\{ {n \atop k} \right\}_{\geq 1} = \left\{ {n \atop k} \right\}.$$

In this paper, we define the restricted and associated Stirling numbers of the first kind by generalizing the (unsigned) Stirling numbers of the first kind, and investigate their basic properties.

2. Associated and restricted Stirling numbers of the first kind

Denote $\left[{n \atop k} \right]$ the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k.$$

In place of the classical (unsigned) Stirling numbers of the first kind $\left[{n \atop k} \right]$, we substitute the (unsigned) restricted Stirling numbers of the first kind $\left[{n \atop k} \right]_{\leq m}$ and the associated (unsigned) Stirling numbers of the first kind $\left[{n \atop k} \right]_{\geq m}$. The associated Stirling number of the first kind $\left[{n \atop k} \right]_{\geq m}$ equals the number of permutations of a set N ($|N| = n$) with k orbits such that each block contains at least m elements ([2, p.256–257], [7]). The restricted Stirling number of the first kind $\left[{n \atop k} \right]_{\leq m}$ equals the number that each block contains at most m elements.

The generating functions of the restricted Stirling numbers of the first kind and the associated Stirling numbers of the first kind are given by the following ([1, p. 467, (12.8)]).

Lemma 1. *For $k \geq 1$ and $m \geq 1$, we have*

$$\sum_{n=0}^{\infty} \left[{n \atop k} \right]_{\leq m} \frac{x^n}{n!} = \frac{1}{k!}\left(x + \frac{x^2}{2} + \cdots + \frac{x^m}{m}\right)^k$$

and

$$\sum_{n=0}^{\infty} \left[{n \atop k} \right]_{\geq m} \frac{x^n}{n!} = \frac{1}{k!}\left(-\log(1-x) - x - \frac{x^2}{2} - \cdots - \frac{x^{m-1}}{m-1}\right)^k.$$

Since the generating function of $\left[{n \atop k} \right]$ is given by

$$\sum_{n=k}^{\infty} \left[{n \atop k} \right] \frac{x^n}{n!} = \frac{(-\log(1-x))^k}{k!},$$

we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\le\infty} = \begin{bmatrix} n \\ k \end{bmatrix}_{\ge 1} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

The unsigned Stirling numbers of the first kind can be calculated by the recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \tag{1}$$

for $k > 0$, with the initial conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \text{ and } \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0$$

for $n > 0$. The restricted and associated Stirling numbers of the first kind can be also calculated by the relations. It is easy to see the initial conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\le m} = 1 \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_{\le m} = \begin{bmatrix} 0 \\ n \end{bmatrix}_{\le m} = 0,$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{\ge m} = 1 \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_{\ge m} = \begin{bmatrix} 0 \\ n \end{bmatrix}_{\ge m} = 0$$

for $n > 0$.

Proposition 1. *For $k > 0$, we have*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\le m} = \sum_{i=0}^{m-1} \frac{n!}{(n-i)!} \begin{bmatrix} n-i \\ k-1 \end{bmatrix}_{\le m},$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\ge m} = \sum_{i=m-1}^{\infty} \frac{n!}{(n-i)!} \begin{bmatrix} n-i \\ k-1 \end{bmatrix}_{\ge m}.$$

Remark. Since

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\le m} = \sum_{i=1}^{n-k+1} \frac{n!}{(n-i)!} \begin{bmatrix} n-i \\ k-1 \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix},$$

both relations in Proposition 1 are reduced to (1), if $m \ge k+2$ and if $m = 1$, respectively.

The classical (unsigned) Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ satisfy the identities:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!,\ \begin{bmatrix} n \\ n \end{bmatrix} = 1,\ \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2},\ \begin{bmatrix} n \\ n-2 \end{bmatrix} = \frac{3n-1}{4}\binom{n}{3},\ \begin{bmatrix} n \\ n-3 \end{bmatrix} = \binom{n}{4}\binom{n}{2}.$$

By the definition or Lemma, we have several basic properties about the restricted Stirling numbers of the first kind.

Lemma 2. *For $0 \le n \le k-1$ or $n \ge km+1$, we have* $\begin{bmatrix} n \\ k \end{bmatrix}_{\le m} = 0$.

For $k \le n \le km$, we have $\begin{bmatrix} n \\ k \end{bmatrix}_{\le m} = \begin{bmatrix} n \\ k \end{bmatrix}$ $(k \le n \le m)$, $\begin{bmatrix} n \\ n \end{bmatrix}_{\le m} = 1$, $\begin{bmatrix} n \\ n-1 \end{bmatrix}_{\le m} = \binom{n}{2}$,

$$\begin{bmatrix} n \\ n-2 \end{bmatrix}_{\le m} = \begin{cases} \frac{3n-1}{4}\binom{n}{3}, & (m \ge 3) \\ 3\binom{n}{4}, & (m = 2) \end{cases}, \quad \begin{bmatrix} n \\ n-3 \end{bmatrix}_{\le m} = \begin{cases} \binom{n}{4}\binom{n}{2}, & (m \ge 4) \\ \frac{5(n+3)}{2}\binom{n}{5}, & (m = 3), \\ 15\binom{n}{6}, & (m = 2) \end{cases}$$

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_{\le m} = (n-1)! \ \ (1 \le n \le m).$$

By the definition or Lemma, we also have several basic properties about the associated Stirling numbers of the first kind.

Lemma 3. *For* $1 \le n \le m-1$, *we have* $\begin{bmatrix} n \\ k \end{bmatrix}_{\ge m} = 0$. *We have*

$$\begin{bmatrix} n \\ n \end{bmatrix}_{\ge m} = \begin{cases} 1, & (m=1) \\ 0, & (m \ge 2) \end{cases}, \quad \begin{bmatrix} n \\ n-1 \end{bmatrix}_{\ge m} = \begin{cases} \binom{n}{2}, & (m=1; m=n=2) \\ 0, & (\text{otherwise}) \end{cases},$$

$$\begin{bmatrix} n \\ n-2 \end{bmatrix}_{\ge m} = \begin{cases} \frac{3n-1}{4}\binom{n}{3}, & (m=1) \\ 3, & (m=2, n=4) \\ 2, & (m=2,3; n=3) \\ 0, & (\text{otherwise}) \end{cases}, \quad \begin{bmatrix} n \\ n-3 \end{bmatrix}_{\ge m} = \begin{cases} \binom{n}{4}\binom{n}{2}, & (m=1) \\ 6, & (2 \le m \le 4, n=4) \\ 20, & (m=2, n=5) \\ 15, & (m-2, n=6) \\ 0, & (\text{otherwise}) \end{cases},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\ge m} = \begin{cases} (n-1)!, & (k=1) \\ 0, & (2 \le k \le m) \end{cases}.$$

References

[1] **Charalambides, C. A.,** Enumerative Combinatorics (Discrete Mathematics and its applications), Chapman and Hall/CRC (2002).

[2] **Comtet, L.**, Advanced Combinatorics, Reidel, Dordrecht (1974).

[3] **Graham, R. L., Knuth, D. E. and Patashnik, O.**, Concrete Mathematics, Second edition, Addison-Wesley, Reading (1994).

[4] **Komatsu, T.**, Hypergeometric Cauchy numbers, *International J. Number Theory*, 9 (2013) 545-546.

[5] **Mező, I.**, Periodicity of the last digits of some combinatorial sequences, *J. Integer Seq.*, 17 (2014) Article 14.1.1.

[6] **Riordan, J.**, Combinatorial identities, John Wiley & Sons, New York (1968).

[7] **Zhao, F.-Z.**, Some properties of associated Strling numbers, *J. Integer Seq.*, 11 (2008) Article 08.1.7.