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Creating subsets of natural numbers with a given weighted density

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ÖSSZEFOGLALÓ. A természetes számok részhalmazainak súlyozott sűrűségeinek vizsgálatakor gyakran dolgozunk $A = \bigcup_{n=1}^{\infty}\big((c_n, d_n] \cap \mathbb{N}\big)$, alakú blokkszerkezetű halmazokkal, ahol (c_n) és (d_n) egész számokból álló sorozatokra teljesül, hogy $0 \le c_n < d_n < c_{n+1}$.
Ebben a cikkben azt vizsgáljuk, hogyan lehet tetszőleges alsó és felső súlyozott sűrűségű halmazokat definiálni, milyen kapcsolat van a blokkok mérete és a súlyozott sűrűség között, és hogyan alkalmazható a Cesàro–Stolz-tétel a súlyozott sűrűségek kiszámítására.
Vizsgáljuk továbbá az $f(n)$ súlyfüggvény és az $f^*(n) = f(n)/(f(1) + \dots + f(n))$ által meghatározott sűrűségek közötti kapcsolatot a blokkszerkezetű halmazok esetében.

ABSTRACT. When studying the weighted densities of subsets of the natural numbers, we often work with block-structured sets of the form $A = \bigcup_{n=1}^{\infty}\big((c_n, d_n] \cap \mathbb{N}\big)$, where the integer sequences (c_n) and (d_n) satisfy $0 \le c_n < d_n < c_{n+1}$.
In this article, we examine how to construct sets with arbitrary lower and upper weighted density, the relationship between the size of the blocks and the weighted density, and how the Cesàro–Stolz theorem can be applied to compute weighted densities.
We also investigate the relationship between densities defined by the weight function $f(n)$ and $f^*(n) = f(n)/(f(1) + \dots + f(n))$ in the case of block-structured sets.

1 Introduction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers. In the quantitative characterization of the size of subsets of natural numbers, one of the most natural and commonly used tools is the *asymptotic density*.

Definition 1. Let $A \subseteq \mathbb{N}$, and denote $A(n) = |\{a \in A : a \le n\}|$. Then the values

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$$

are called the lower and upper asymptotic density of the set A, respectively. If $\underline{d}(A) = \overline{d}(A)$, then this common value is the asymptotic density of the set A [6].

HUNGARIAN TITLE. A természetes számok adott súlyozott sűrűségű részhalmazainak konstruálása.
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Denote by χ_A the characteristic function of the set A, that is

$$\chi_A(n) = \begin{cases} 1, & \text{if} \quad n \in A; \\ 0, & \text{if} \quad n \notin A. \end{cases}$$

We call a function $f\colon \mathbb{N} \to \mathbb{R}^+$ an *Erdős–Ulam function* if it satisfies $f(1) = 1$,

$$\sum_{n=1}^{\infty} f(n) = \infty,$$

and

$$\lim_{n\to\infty} f^*(n) = 0, \quad \text{where} \quad f^*(n) = \frac{f(n)}{\sum\limits_{j=1}^{n} f(j)}\,. \tag{1}$$

Using an Erdős–Ulam function, we can define an *Erdős–Ulam density* as follows.

Let f be an Erdős–Ulam function. For any $A \subset \mathbb{N}$, let

$$F_A(n) = \frac{A_f(n)}{\mathbb{N}_f(n)}, \quad \text{where} \quad A_f(n) = \sum_{j=1}^{n} f(j) \cdot \chi_A(j), \quad \mathbb{N}_f(n) = \sum_{j=1}^{n} f(j) \quad [2].$$

Obviously, for every $n \in \mathbb{N}$, $0 \le F_A(n) \le 1$.

Let

$$\underline{d_f}(A) = \liminf_{n\to\infty} F_A(n) \quad \text{and} \quad \overline{d_f}(A) = \limsup_{n\to\infty} F_A(n)$$

be the lower and upper f-density of the set A [1]. It is obvious that

$$0 \le \underline{d_f}(A) \le \overline{d_f}(A) \le 1. \tag{2}$$

If $\underline{d_f}(A) = \overline{d_f}(A)$, then the set A *possesses* an f-density, which we denote by $d_f(A)$.

The asymptotic density corresponds to the choice $f(n) = 1$, while the logarithmic density corresponds to the choice $f(n) = \frac{1}{n}$.

For any Erdős–Ulam function f, the f-density of finite sets is 0, while that of sets with finite complement is 1. The other subsets of natural numbers can be written in the following form [3, 4].

Definition 2. Let (c_n) and (d_n) be sequences of integers such that

$$0 \le c_n < d_n < c_{n+1}.$$

Let

$$A = \bigcup_{n=1}^{\infty} \big((c_n, d_n] \cap \mathbb{N}\big). \tag{3}$$

The Erdős–Ulam density of certain sets given in the form (3) can be computed using the Cesàro–Stolz theorem [5].

Cesàro–Stolz theorem. Let (a_m) and (b_m) be real sequences such that

(i) $0 < b_1 < b_2 < \dots$ and $\lim_{m\to\infty} b_m = \infty$,

(ii) $\lim_{m\to\infty} \frac{a_{m+1} - a_m}{b_{m+1} - b_m} = k \in \mathbb{R}$.

Then $\lim_{m\to\infty} \frac{a_m}{b_m} = k$.

If we substitute sequences of the form $a_m = \sum_{n=1}^{m} \alpha_n, \quad b_m = \sum_{n=1}^{m} \beta_n$ into the theorem, we obtain the following theorem which is more suitable for us.

Corollary 3. *(Cesàro–Stolz theorem [5, page 42].) Let $(\alpha_n), (\beta_n)$ be sequences of positive real numbers such that*

$$\sum_{n=1}^{\infty} \beta_n = \infty,$$

and

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = k.$$

Then

$$\lim_{m\to\infty} \frac{\sum_{n=1}^{m} \alpha_n}{\sum_{n=1}^{m} \beta_n} = k.$$

The following two lemmas present the property of sets of the form (3) that the index subsequence belonging to the limit inferior of the sequence $\left(\frac{A_f(c_n)}{\mathbb{N}_f(c_n)}\right)$ is a subsequence of the (c_n) sequence, and the index subsequence belonging to the limit superior is a subsequence of the (d_n) sequence.

Lemma 4. *Let $A \subseteq \mathbb{N}$. Then for every natural number $n \geq 2$*

- *if $n \in A$, then* $\dfrac{A_f(n)}{\mathbb{N}_f(n)} \geq \dfrac{A_f(n-1)}{\mathbb{N}_f(n-1)}$,
- *if $n \notin A$, then* $\dfrac{A_f(n)}{\mathbb{N}_f(n)} \leq \dfrac{A_f(n-1)}{\mathbb{N}_f(n-1)}$.

Proof. Let $n \geq 2$ and $n \in A$. Then $A_f(n) = A_f(n-1) + f(n)$, therefore

$$\frac{A_f(n)}{\mathbb{N}_f(n)} = \frac{A_f(n-1) + f(n)}{\mathbb{N}_f(n-1) + f(n)} \geq \frac{A_f(n-1)}{\mathbb{N}_f(n-1)}.$$

Now let $n \geq 2$ and $n \notin A$. Then $A_f(n) = A_f(n-1)$, therefore

$$\frac{A_f(n)}{\mathbb{N}_f(n)} = \frac{A_f(n-1)}{\mathbb{N}_f(n)} \leq \frac{A_f(n-1)}{\mathbb{N}_f(n-1)}.$$

□

Corollary 5. *Let A be a set defined in the way (3). Then*

$$\underline{d_f}(A) = \liminf_{n\to\infty} \frac{A_f(c_n)}{\mathbb{N}_f(c_n)} \quad \textit{and} \quad \overline{d_f}(A) = \limsup_{n\to\infty} \frac{A_f(d_n)}{\mathbb{N}_f(d_n)}.$$

2 Results

In the following theorem, we give a condition under which the index sequence belonging to the limit inferior of the sequence $\left(\frac{A_f(c_n)}{\mathbb{N}_f(c_n)}\right)$, respectively to the limit superior, coincides with the (c_n), respectively (d_n) sequences.

Theorem 6. *Let A be a set defined in the way* (3)*. If*

$$\left[\lim_{n\to\infty}(\mathbb{N}_f(c_{n+1})-\mathbb{N}_f(c_n))=\infty \quad \textit{and} \quad \lim_{n\to\infty}\frac{\mathbb{N}_f(d_n)-\mathbb{N}_f(c_n)}{\mathbb{N}_f(c_{n+1})-\mathbb{N}_f(c_n)}=\alpha,\right]$$
$$\left(\lim_{n\to\infty}(\mathbb{N}_f(d_{n+1})-\mathbb{N}_f(d_n))=\infty \quad \textit{and} \quad \lim_{n\to\infty}\frac{\mathbb{N}_f(d_{n+1})-\mathbb{N}_f(c_{n+1})}{\mathbb{N}_f(d_{n+1})-\mathbb{N}_f(d_n)}=\beta,\right)$$

then

$$\left[\underline{d_f}(A)=\lim_{n\to\infty}\frac{A_f(c_n)}{\mathbb{N}_f(c_n)}=\alpha,\right]$$
$$\left(\overline{d_f}(A)=\lim_{n\to\infty}\frac{A_f(d_n)}{\mathbb{N}_f(d_n)}=\beta\right).$$

Proof. The statement can be proved directly using the Cesàro–Stolz theorem. Define the

$$\beta_n=\mathbb{N}_f(c_{n+1})-\mathbb{N}_f(c_n),\quad \beta'_n=\mathbb{N}_f(d_{n+1})-\mathbb{N}_f(d_n),$$
$$\alpha_n=\mathbb{N}_f(d_n)-\mathbb{N}_f(c_n),\quad \alpha'_n=\mathbb{N}_f(d_{n+1})-\mathbb{N}_f(c_{n+1})$$

sequences. Apply the Cesàro–Stolz theorem to the sequences $(\alpha_n), (\beta_n)$, and $(\alpha'_n), (\beta'_n)$, from which the statement follows immediately. □

Remark 7. If

$$\lim_{n\to\infty}\frac{\mathbb{N}_f(c_{n+1})-\mathbb{N}_f(c_n)}{\mathbb{N}_f(d_n)-\mathbb{N}_f(d_{n-1})}=1,$$

then in the previous theorem $\alpha=\beta$, that is

$$\underline{d_f}(A)=\overline{d_f}(A).$$

In the following theorem, we provide a method for constructing sequences (c_n) and (d_n) that define a set with a prescribed lower f-density α and upper f-density β.

Theorem 8. *Let f be a non-increasing Erdős–Ulam function and $0\le\alpha<\beta\le 1$. Define the set*

$$A=\bigcup_{n=1}^{\infty}\left((c_n,d_n]\cap\mathbb{N}\right),$$

where the sequences (c_n) and (d_n) are constructed as follows. Let $i\colon\mathbb{N}\to\mathbb{R}$ be a function such that $\lim_{n\to\infty} i(n)=\infty$ (e.g., $i(n)=n$). Let c_1 be an arbitrary positive integer. For each $n\ge 1$:

- *If $\beta<1$, then*
$$d_n=\min\left\{k\in\mathbb{N}:\mathbb{N}_f(k)>\frac{1-\alpha}{1-\beta}\,\mathbb{N}_f(c_n)\right\}. \tag{4}$$
- *If $\beta=1$, then*
$$d_n=\min\left\{k\in\mathbb{N}:\mathbb{N}_f(k)>i(n)\,\mathbb{N}_f(c_n)\right\}. \tag{5}$$
- *If $\alpha>0$, then*
$$c_{n+1}=\min\left\{k\in\mathbb{N}:\mathbb{N}_f(k)>\frac{\beta}{\alpha}\,\mathbb{N}_f(d_n)\right\}. \tag{6}$$
- *If $\alpha=0$, then*
$$c_{n+1}=\min\left\{k\in\mathbb{N}:\mathbb{N}_f(k)>i(n)\,\mathbb{N}_f(d_n)\right\}. \tag{7}$$

Then, the set A satisfies

$$\underline{d_f}(A) = \alpha \quad \textit{and} \quad \overline{d_f}(A) = \beta.$$

Proof. We proceed by considering the different cases based on the values of α and β.

- Case $\beta < 1$. By (4)
$$\frac{1-\alpha}{1-\beta}\mathbb{N}_f(c_n) < \mathbb{N}_f(d_n) \leq \frac{1-\alpha}{1-\beta}\mathbb{N}_f(c_n) + f(d_n+1).$$
Since (1) and f is non-increasing
$$\mathbb{N}_f(d_n) = \frac{1-\alpha}{1-\beta}\mathbb{N}_f(c_n) + o(\mathbb{N}_f(c_n)). \tag{8}$$
  - Subcase $\alpha > 0$ (then $\beta > 0$). By (6)
$$\frac{\beta}{\alpha}\mathbb{N}_f(d_n) < \mathbb{N}_f(c_{n+1}) \leq \frac{\beta}{\alpha}\mathbb{N}_f(d_n) + f(c_{n+1}+1),$$
giving
$$\mathbb{N}_f(c_{n+1}) = \frac{\beta}{\alpha}\mathbb{N}_f(d_n) + o(\mathbb{N}_f(d_n)). \tag{9}$$
From (8) and (9)
$$\begin{aligned}\mathbb{N}_f(d_{n+1}) &= \frac{1-\alpha}{1-\beta}\mathbb{N}_f(c_{n+1}) + o(\mathbb{N}_f(c_{n+1}))\\ &= \frac{1-\alpha}{1-\beta}\cdot\frac{\beta}{\alpha}\mathbb{N}_f(d_n) + o(\mathbb{N}_f(d_n)),\\ \mathbb{N}_f(c_{n+1}) &= \frac{\beta}{\alpha}\cdot\frac{1-\alpha}{1-\beta}\mathbb{N}_f(c_n) + o(\mathbb{N}_f(c_n)).\end{aligned}$$
Then
$$\lim_{n\to\infty}\frac{\mathbb{N}_f(d_{n+1}) - \mathbb{N}_f(c_{n+1})}{\mathbb{N}_f(d_{n+1}) - \mathbb{N}_f(d_n)} = \lim_{n\to\infty}\frac{\frac{1-\alpha}{1-\beta}\frac{\beta}{\alpha} - \frac{\beta}{\alpha} + o(1)}{\frac{1-\alpha}{1-\beta}\frac{\beta}{\alpha} - 1 + o(1)} = \beta,$$
and
$$\lim_{n\to\infty}\frac{\mathbb{N}_f(d_n) - \mathbb{N}_f(c_n)}{\mathbb{N}_f(c_{n+1}) - \mathbb{N}_f(c_n)} = \lim_{n\to\infty}\frac{\frac{1-\alpha}{1-\beta} - 1 + o(1)}{\frac{1-\alpha}{1-\beta}\cdot\frac{\beta}{\alpha} - 1 + o(1)} = \alpha.$$
  - Subcase $\alpha = 0$. By (7)
$$\lim_{n\to\infty}\frac{\mathbb{N}_f(d_n)}{\mathbb{N}_f(c_{n+1})} = 0.$$
Then
$$\lim_{n\to\infty}\frac{\mathbb{N}_f(d_n) - \mathbb{N}_f(c_n)}{\mathbb{N}_f(c_{n+1}) - \mathbb{N}_f(c_n)} = \lim_{n\to\infty}\frac{\frac{\mathbb{N}_f(d_n)}{\mathbb{N}_f(c_{n+1})} - \frac{\mathbb{N}_f(c_n)}{\mathbb{N}_f(c_{n+1})}}{1 - \frac{\mathbb{N}_f(c_n)}{\mathbb{N}_f(c_{n+1})}} = 0. \tag{10}$$
From (8) for $n+1$
$$\begin{aligned}&\lim_{n\to\infty}\frac{\mathbb{N}_f(d_{n+1}) - \mathbb{N}_f(c_{n+1})}{\mathbb{N}_f(d_{n+1}) - \mathbb{N}_f(d_n)}\\ &= \lim_{n\to\infty}\frac{\frac{\mathbb{N}_f(d_{n+1})}{\mathbb{N}_f(c_{n+1})} - 1}{\frac{\mathbb{N}_f(d_{n+1})}{\mathbb{N}_f(c_{n+1})} - \frac{\mathbb{N}_f(d_n)}{\mathbb{N}_f(c_{n+1})}}\\ &= \lim_{n\to\infty}\frac{\frac{1-\alpha}{1-\beta} - 1 + o(1)}{\frac{1-\alpha}{1-\beta} - \frac{\mathbb{N}_f(d_n)}{\mathbb{N}_f(c_{n+1})} + o(1)} = \beta.\end{aligned}$$

- Case $\beta = 1$. By (5)
$$\lim_{n\to\infty} \frac{\mathbb{N}_f(c_n)}{\mathbb{N}_f(d_n)} = 0.$$
Then
$$\lim_{n\to\infty} \frac{\mathbb{N}_f(d_{n+1}) - \mathbb{N}_f(c_{n+1})}{\mathbb{N}_f(d_{n+1}) - \mathbb{N}_f(d_n)} = \lim_{n\to\infty} \frac{1 - \frac{\mathbb{N}_f(c_{n+1})}{\mathbb{N}_f(d_{n+1})}}{1 - \frac{\mathbb{N}_f(d_n)}{\mathbb{N}_f(d_{n+1})}} = 1.$$
  - Subcase $\alpha > 0$. From (6)
  $$\mathbb{N}_f(c_{n+1}) = \frac{\beta}{\alpha}\mathbb{N}_f(d_n) + o(\mathbb{N}_f(d_n)).$$
  Then
  $$\begin{aligned}&\lim_{n\to\infty} \frac{\mathbb{N}_f(d_n) - \mathbb{N}_f(c_n)}{\mathbb{N}_f(c_{n+1}) - \mathbb{N}_f(c_n)} \\ &= \lim_{n\to\infty} \frac{1 - \frac{\mathbb{N}_f(c_n)}{\mathbb{N}_f(d_n)}}{\frac{\mathbb{N}_f(c_{n+1})}{\mathbb{N}_f(d_n)} - \frac{\mathbb{N}_f(c_n)}{\mathbb{N}_f(d_n)}} \\ &= \lim_{n\to\infty} \frac{1 - \frac{\mathbb{N}_f(c_n)}{\mathbb{N}_f(d_n)}}{\frac{\beta}{\alpha} - \frac{\mathbb{N}_f(c_n)}{\mathbb{N}_f(d_n)} + o(1)} = \alpha.\end{aligned}$$
  - Subcase $\alpha = 0$. Then (10) holds.

In all cases, the conditions of Theorem 6 are satisfied, completing the proof. □

In the following lemma, we give a precise condition for the existence of f-density.

Lemma 9. *Let $A \subseteq \mathbb{N}$ and f be an Erdős–Ulam function. The set A has f-density exactly when there exists a sequence (i_n) of natural numbers such that*

(i) *for every $n \in \mathbb{N}$, $i_n < i_{n+1}$,*

(ii) $\liminf_{n\to\infty} \frac{A_f(n)}{\mathbb{N}_f(n)} = \lim_{n\to\infty} \frac{A_f(i_{2n-1})}{\mathbb{N}_f(i_{2n-1})}$,

(iii) $\limsup_{n\to\infty} \frac{A_f(n)}{\mathbb{N}_f(n)} = \lim_{n\to\infty} \frac{A_f(i_{2n})}{\mathbb{N}_f(i_{2n})}$,

(iv) $\lim_{n\to\infty} \frac{\mathbb{N}_f(i_{n+1})}{\mathbb{N}_f(i_n)} = 1$.

Proof. If $\underline{d_f}(A) = \overline{d_f}(A)$, then let the sequence $i_n = n$ $(n = 1, 2, \dots)$. Then properties (i)-(iv) obviously hold.

Let (i_n) be such a sequence that properties (i)-(iv) hold
$$\overline{d_f}(A) = \limsup_{n\to\infty} \frac{A_f(n)}{\mathbb{N}_f(n)} = \lim_{n\to\infty} \frac{A_f(i_{2n})}{\mathbb{N}_f(i_{2n})} \le \liminf_{n\to\infty} \frac{A_f(i_{2n-1}) + \mathbb{N}_f(i_{2n}) - \mathbb{N}_f(i_{2n-1})}{\mathbb{N}_f(i_{2n})}$$
$$\le \liminf_{n\to\infty} \left(\frac{A_f(i_{2n-1})}{\mathbb{N}_f(i_{2n-1})} + 1 - \frac{\mathbb{N}_f(i_{2n-1})}{\mathbb{N}_f(i_{2n})}\right) = \underline{d_f}(A).$$
From which it follows that $\underline{d_f}(A) = \overline{d_f}(A)$. □

In the following, we examine what value the lower and upper f-density takes for sets of the form (3), if it does not have f^*-density.

Lemma 10. *Let f be an Erdős–Ulam function. If for some sequence of integers (i_n)*

$$\liminf_{n\to\infty} \frac{\mathbb{N}_{f^*}(i_{n+1})}{\mathbb{N}_{f^*}(i_n)} > 1,$$

then

$$\lim_{n\to\infty} \frac{\mathbb{N}_f(i_{n+1})}{\mathbb{N}_f(i_n)} = \infty.$$

Proof. We can write the following inequality.

$$\begin{aligned}\mathbb{N}_{f^*}(i_{n+1}) &= \mathbb{N}_{f^*}(i_n) + \sum_{k=i_n+1}^{i_{n+1}} \frac{f(k)}{\mathbb{N}_f(k)} \\ &\le \mathbb{N}_{f^*}(i_n) + \sum_{k=i_n+1}^{i_{n+1}} \frac{f(k)}{\mathbb{N}_f(i_n)} \\ &= \mathbb{N}_{f^*}(i_n) + \frac{\mathbb{N}_f(i_{n+1}) - \mathbb{N}_f(i_n)}{\mathbb{N}_f(i_n)}.\end{aligned}$$

From this, it directly follows that

$$\frac{\mathbb{N}_f(i_{n+1})}{\mathbb{N}_f(i_n)} \ge \mathbb{N}_{f^*}(i_{n+1}) - \mathbb{N}_{f^*}(i_n). \tag{11}$$

According to the condition, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\frac{\mathbb{N}_{f^*}(i_{m+1})}{\mathbb{N}_{f^*}(i_m)} \ge 1 + \delta \quad \text{for every} \quad m > n_0.$$

From this it follows that

$$\mathbb{N}_{f^*}(i_{m+1}) - \mathbb{N}_{f^*}(i_m) \ge \delta \mathbb{N}_{f^*}(i_m).$$

According to inequality (11)

$$\frac{\mathbb{N}_f(i_{m+1})}{\mathbb{N}_f(i_m)} \ge \mathbb{N}_{f^*}(i_{m+1}) - \mathbb{N}_{f^*}(i_m) \ge \delta \mathbb{N}_{f^*}(i_m).$$

Since $\mathbb{N}_{f^*}(i_m) \to \infty$ (as $m \to \infty$), therefore

$$\limsup_{n\to\infty} \frac{\mathbb{N}_f(i_{n+1})}{\mathbb{N}_f(i_n)} = \infty.$$

□

Theorem 11. *Let f be an Erdős–Ulam function, and let (i_n) be a sequence of integers such that*

$$\liminf_{n\to\infty} \frac{\mathbb{N}_{f^*}(i_{n+1})}{\mathbb{N}_{f^*}(i_n)} > 1. \tag{12}$$

Let $L, H \subseteq \mathbb{N}$ be sets that possess f-density and satisfy $d_f(L) < d_f(H)$. Define

$$A = \bigcup_{n=1}^{\infty} \Big(\big((i_{2n-1}, i_{2n}] \cap H \big) \cup \big((i_{2n}, i_{2n+1}] \cap L \big) \Big).$$

Then

$$\underline{d_f}(A) = d_f(L), \qquad \overline{d_f}(A) = d_f(H).$$

Proof. Based on the definition of set A

$$A_f(i_{2n}) - A_f(i_{2n-1}) = H_f(i_{2n}) - H_f(i_{2n-1}).$$

According to Corollary 5

$$\begin{aligned}\overline{d_f}(A) &= \limsup_{n\to\infty} \frac{A_f(i_{2n})}{\mathbb{N}_f(i_{2n})} \\ &= \limsup_{n\to\infty} \frac{A_f(i_{2n-1}) + H_f(i_{2n}) - H_f(i_{2n-1})}{\mathbb{N}_f(i_{2n})} \\ &= \limsup_{n\to\infty} \left(\frac{A_f(i_{2n-1})}{\mathbb{N}_f(i_{2n-1})} \frac{\mathbb{N}_f(i_{2n-1})}{\mathbb{N}_f(i_{2n})} + \frac{H_f(i_{2n})}{\mathbb{N}_f(i_{2n})} - \frac{H_f(i_{2n-1})}{\mathbb{N}_f(i_{2n-1})} \frac{\mathbb{N}_f(i_{2n-1})}{\mathbb{N}_f(i_{2n})} \right).\end{aligned}$$

Based on (12) and Lemma 10

$$\lim_{n\to\infty} \frac{\mathbb{N}_f(i_{n-1})}{\mathbb{N}_f(i_n)} = 0.$$

Hence, it follows that

$$\overline{d_f}(A) = \lim_{n\to\infty} \frac{H_f(i_{2n})}{\mathbb{N}_f(i_{2n})} = d_f(H).$$

Similarly, it can be proven that

$$\underline{d_f}(A) = \lim_{n\to\infty} \frac{A_f(i_{2n+1})}{\mathbb{N}_f(i_{2n+1})} = d_f(L).$$

□

3 Conclusion

In this article, we examined certain properties of sets of the form (3) with respect to Erdős–Ulam densities.

In Theorem 6, we determined the f-densities of sets defined by sequences (c_n) and (d_n), and provided a condition for the existence of the f-density. Using Theorem 6, we then defined a method that constructs a set $A \subseteq \mathbb{N}$ for any $0 \le \alpha < \beta \le 1$ such that $\underline{d_f}(A) = \alpha$ and $\overline{d_f}(A) = \beta$.

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