


From tiling to matching: exploring Fibonacci cube structures

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ÖSSZEFOGLALÓ. Számos publikáció foglalkozik különböző alakú síkidomokkal való elhelyezésekkel vagy a kombinatorika egyik nagyon érdekes alakzatának vizsgálatával, a Fibonacci-kocka gráffal. Mi ezt a két kutatási területet kombináljuk. Rövid cikkünkben egy bijektív megfelelést adunk az ún. Fibonacci-elhelyezések és a Fibonacci-kockák csúcsai között. Meghatározzuk az azonos típusú mozaikok közötti egyértelmű szomszédságot, amely megadja a megfelelő gráf éleit. Ezenkívül példákat mutatunk be a cikkcakk alakú hatszögletű láncok és a Fibonacci-kocka teljes párosításainak kapcsolatára.

ABSTRACT. Several articles deal with tilings with various shapes, and also a very frequent type of combinatorics is to examine the Fibonacci cube graph and its generalizations. We combine these two things. In our note, we give a one-to-one contention between the Fibonacci tilings and vertices of the Fibonacci cubes. We define a clear adjacency between the tiles of the same type, which gives us the edges of the corresponding graph. In addition, we present examples of the relationship between perfect matchings of zigzag hexagonal chains and Fibonacci cube.

1 Introduction

Hundreds, if not thousands, of articles can be found about tilings and about the Fibonacci cube, but to the best of our knowledge, there is not any paper that deals with both at the same time. In this article, we connect these two subfields of combinatorics as follows. We define a neighborhood relationship among the tilings, which allows us to represent the tilings in a graph structure that is isomorphic to the Fibonacci cube. It is important to look for connections between different areas of mathematics, as it can be beneficial to each area. The results having achieved in one of the fields may be applied and reinterpreted in the other field, or fields, which can give new impetus to research. We hope that our observations will be useful in the combinatorial study of tilings and the Fibonacci cube as well.

Let t_n be the number of the different tilings (n -tilings) with (1×1) -squares and (1×2) -dominoes (two squares with a common edge) of a $(1 \times n)$ -board. Hereafter, a square always means (1×1) -square and a domino means (1×2) -domino. It is known, e.g., in [3], that the number of possible tilings of a $(1 \times n)$ -board is $t_n = F_{n+1}$, where (F_n) denotes the Fibonacci

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sequence (defined by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ and A000045 in OEIS [15]). Some researchers generalized the tilings with colored or weighted shapes, i.e., [1, 4, 16]. Moreover, Komatsu, Németh, and Szalay [12] examined the tilings with colored squares and dominoes on the hyperbolic $(2 \times n)$ -board, and they gave the fourth order linear homogeneous recurrence relation of t_n , where the coefficients satisfy recurrence sequences, as well. There are numerous articles in which authors examine various generalizations or restrictions, i.e., with regard to shapes, colors, or dimensions. For example, Khadir, Németh, and Szalay [9] examined a type of tilings in which the colors had a specific order and a “darker” element could never come before a “lighter” element.

The Fibonacci cube of dimension $n \geq 1$, denoted as Γ_n , is the subgraph of the hypercube Q_n induced by vertices with no consecutive 1s in their coordinates. Hsu [8] introduced this graph as an interconnection network. Γ_n is an isometric subgraph and is inspired by the Fibonacci numbers. It has attractive recurrent structures, such as its decomposition into two subgraphs which are also Fibonacci cubes themselves. Structural properties of these graphs were more extensively studied afterwards; see, for example, the survey [10] and the recent book [7]. There are several kinds of cube graphs, ex., the Lucas cubes Λ_n , introduced in [14], that have attracted attention as well due to the fact that these cubes are the cyclic version of Fibonacci cubes. For other generalizations, i.e., see [2, 6]. They have also been widely studied.

In this article, we combine both of the above-mentioned combinatorial examinations.

2 Structure of tilings with squares and dominoes

In this section, we define a neighbouring among the n -tilings, which implies a structure.

Let the *divide-join base transformation*, shortly divide-join transform, be when we divide a domino into two squares or we join two consecutive squares into one domino (see: $\square \leftrightarrow \square\square$). We say, that two n -tilings are *neighboring* if we can transform them into each other with only one divide-join transform. For example, in Figure 1, the two 6-tilings are neighbors, they differ exclusively in the light blue parts.



Figure 1. Two neighbouring tilings

Let Γ_n for $n \geq 1$ be the graph, where the vertices are the $(n + 1)$ -tilings and between any two neighboring tilings there is an edge. For example, Figure 2 shows the graph Γ_3 .

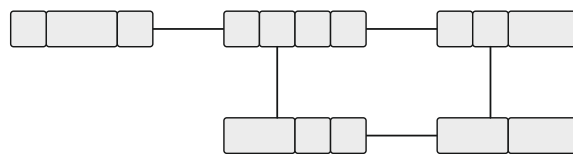


Figure 2. Graph Γ_3

Theorem 1. For $n \geq 1$ the graph Γ_n is isomorphic to the Fibonacci n -cube Γ_n .

Proof. Now we give a one-to-one correspondence between $(1 \times n)$ -tilings and the binary bit strings of length $n - 1$ with no block of 2 or more 1s. The base of the bijection is what follows. Imagine a $(1 \times n)$ -square grid behind the tiling, where neighbor squares are separated from each other by vertical edges. If such an edge is covered by the tiling, then we write 1 above

the edge; otherwise, we write 0. Figure 3 shows all the $(1 \times n)$ -tilings and their corresponding binary strings for $n = 2, 3, 4, 5$. (We mention that in [9], there was introduced a similar code for a type of general tilings on $(1 \times n)$ -board, but the rules of 0 and 1 were changed.) Because we tile with at most dominoes, there is no two consecutive 1s in any codes which are the vertices of graph Γ_n . Moreover, in case of the neighboring tilings, the codes differ only one bit. That means, the corresponding vertices are joined with edges in Γ_n and in n -cube Γ_n as well. \square

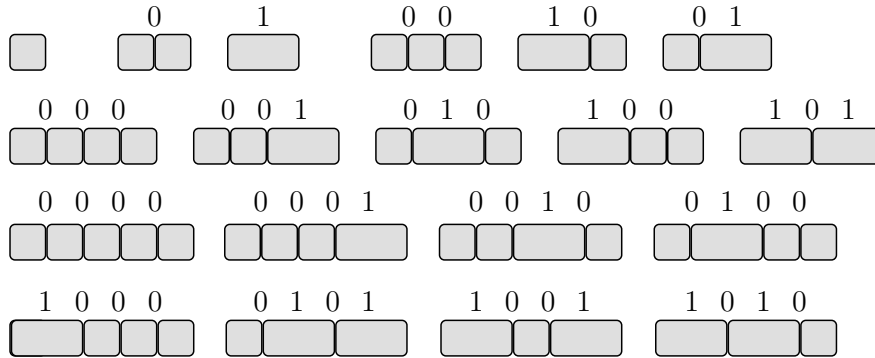


Figure 3. All the n -tilings and their codes up to $n = 5$

Figures 4 and 5 show the structures of some n -tilings in graph forms.

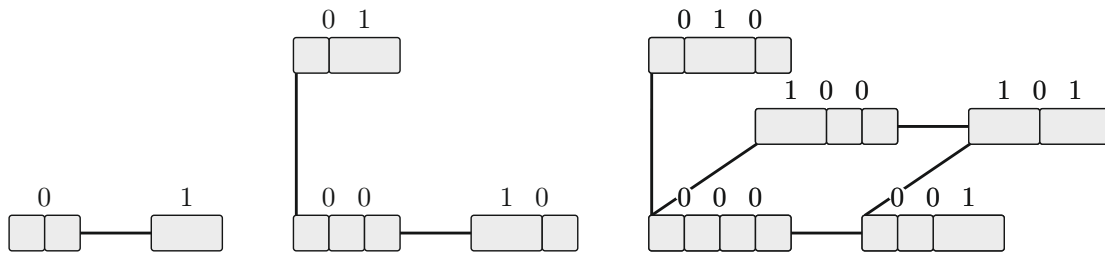


Figure 4. Fibonacci cubes with tiled vertices if $n = 1, 2, 3$

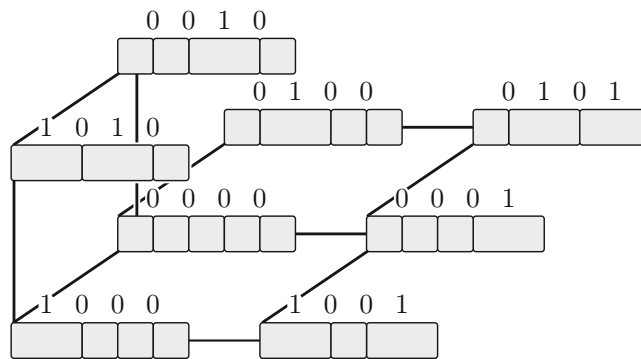


Figure 5. Fibonacci cube with tiled vertices when $n = 4$

2.1 Concluding remarks and future works

2.1.1 Fibonacci cubes and tilings

The connection of the tilings and the Fibonacci cubes can provide many new theorems. For example:

- It is well known that every Fibonacci cube has a Hamiltonian path. That means that all n -tilings can be drawn in a sequence, where the consecutive tiling are neighbours. Figure 6 shows a path, moreover, with the dashed line a circle of 5-tilings.

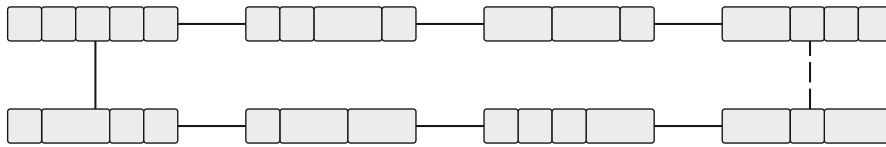


Figure 6. Neighbouring circle of the 5-tilings

- By the omitting the tilings, where the first and last peaces are dominoes (their codes in the first and last positions contain 1s) the structure of the n -tilings is isomorphic to the Lucas cubes Λ_{n-1} (see Figure 7).

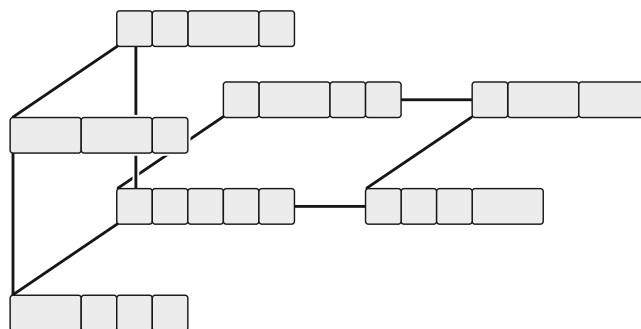


Figure 7. Lucas cube

- The presented method allows for the definition of additional tiling structures, such as tribonacci cubes [2], Pell graphs [13].

2.1.2 Resonance graphs – graphs of perfect matchings

It is well known that Fibonacci cubes are the resonance graphs of fibonacenes which are specific types of hexagonal chains, where no three consecutive hexagons are in a straight line; see the paper by Klavzar and Zigert [11]. On the other hand, it is easy to construct a bijection between perfect matchings in a zig-zag fibonacene with n hexagons and all matchings in a path of length n . An example is given in Fig. 3 of Doslic’s paper [5], where the path and the corresponding matchings are obtained by taking the horizontal edges in the backbone path of a fibonacene. A matching is a set of edges in a graph, where no two edges share a common vertex, and a matching is perfect when every vertex in the graph is an endpoint of an edge in the matching.

Using the notations, where the chosen edges of hexagonal chains are double lines (see [5]) we show the connection among the perfect matchings of zigzag hexagonal chains, Fibonacci tilings, and binary strings in Figure 8. (A small circle appears inside a hexagonal, if it has three edges in the perfect matching.) The key links are the blue edges from the “backbone” of the perfect matching as we see in the figure. Each horizontal blue line is covered by a domino.

Thus, the vertices of the resonance graphs are perfect matchings and two perfect matchings are connected in the resonance graph if their symmetric difference (the set of edges in one but not the other) forms a cycle around a hexagonal. In Figure 9, the two perfect matchings are neighbors, they connect along the blue hexagons. (The symmetrical difference of these is a hexagon.)

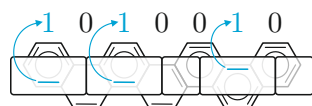


Figure 8. Connection among the perfect matchings of zigzag hexagonal chains, Fibonacci tilings, and binary strings



Figure 9. Two neighbouring perfect matchings

Figures 10 and 11 show the resonance graphs with corresponding binary strings when $n = 1, 2, 3, \text{ or } 4$.

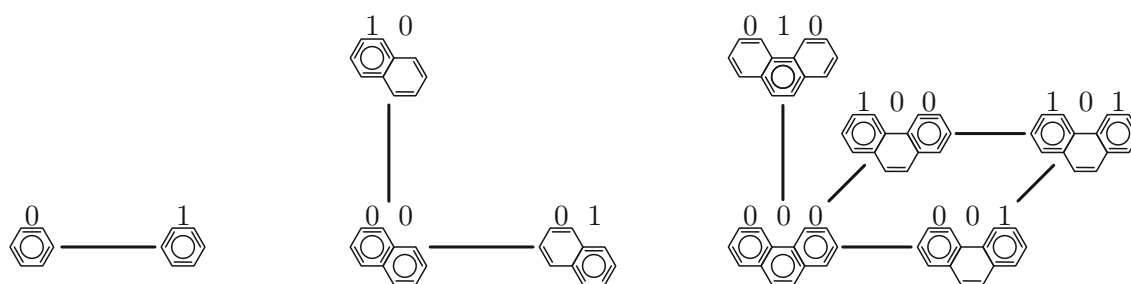


Figure 10. Fibonacci cubes with perfect matching vertices if $n = 1, 2, 3$

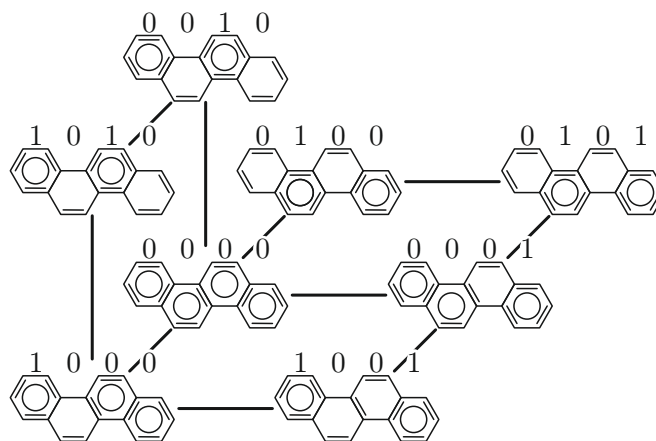


Figure 11. Fibonacci cubes with perfect matching vertices if $n = 4$

Now let's examine all the matching and their graph structures, not just the perfect matchings for $n = 1$ and 2. Let two matchings be neighbors if their symmetric differential is a single edge. (The two matchings differ in only one edge.) Then Figure 12 shows the graph structure of all 18 matchings when $n = 1$. As we see the graph is a connected subgraph of the 3-dimensional cube mosaic graph. The blue edges form two cubes that share a common vertex, which is the empty (edge-less) matching. These two cube graphs together can be an extension of the corresponding resonance graph, see left subfigure of Figure 10.

Similarly, if we expand the resonance graph in the case of $n = 2$, we obtain Figure 13. It contains for cube graphs with a common edges.

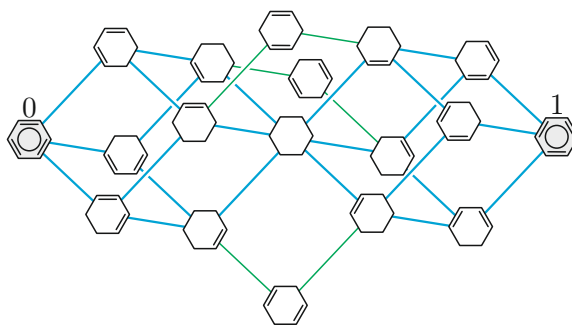


Figure 12. Graph of all matchings of zigzag hexagonal chain if $n = 1$

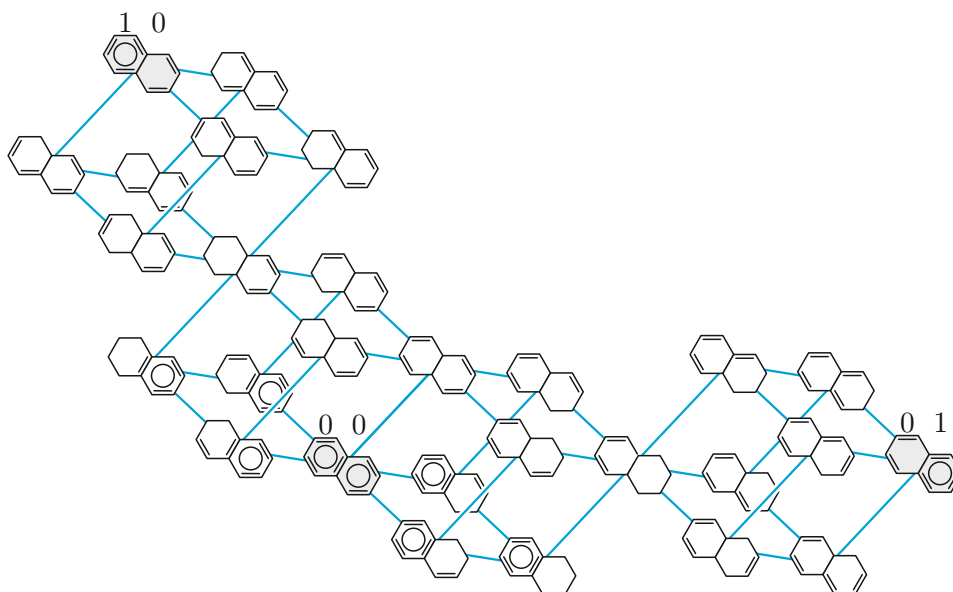



Figure 13. An extension graph of resonance graph when $n = 2$

It is obvious that any edge of a resonance graph can be extended in a similar way, only the connecting hexagonal parts change. Thus, we can formulate the following statement.

Theorem 2. *All the edges of a resonance graphs can be extended by two 3-dimensional cube graphs which vertices are matchings.*

But what might the complete graph structure look like? We present Figure 14 which contain some matchings when $n = 2$. This graph is also a subgraph of the 3-dimensional cube mosaic graph, but the structure of all the 148 matchings cannot be a subgraph of the 3-dimensional cube mosaic graph, because, for example, the matching without edges () has 11 neighbours. Therefore, this part (the graph of the empty matching and its neighbours) is at least a subgraph of a 6-dimensional hypercube mosaic. Based on our observations, we can conclude that most of the structure may be a subgraph of the cube- or a higher-dimensional hypercube mosaics. And if the graph structure of all the matchings of the n -long zigzag hexagonal chain is a subgraph of the d -dimensional hypercube mosaic, than d is $\lceil \frac{5n+1}{2} \rceil$.

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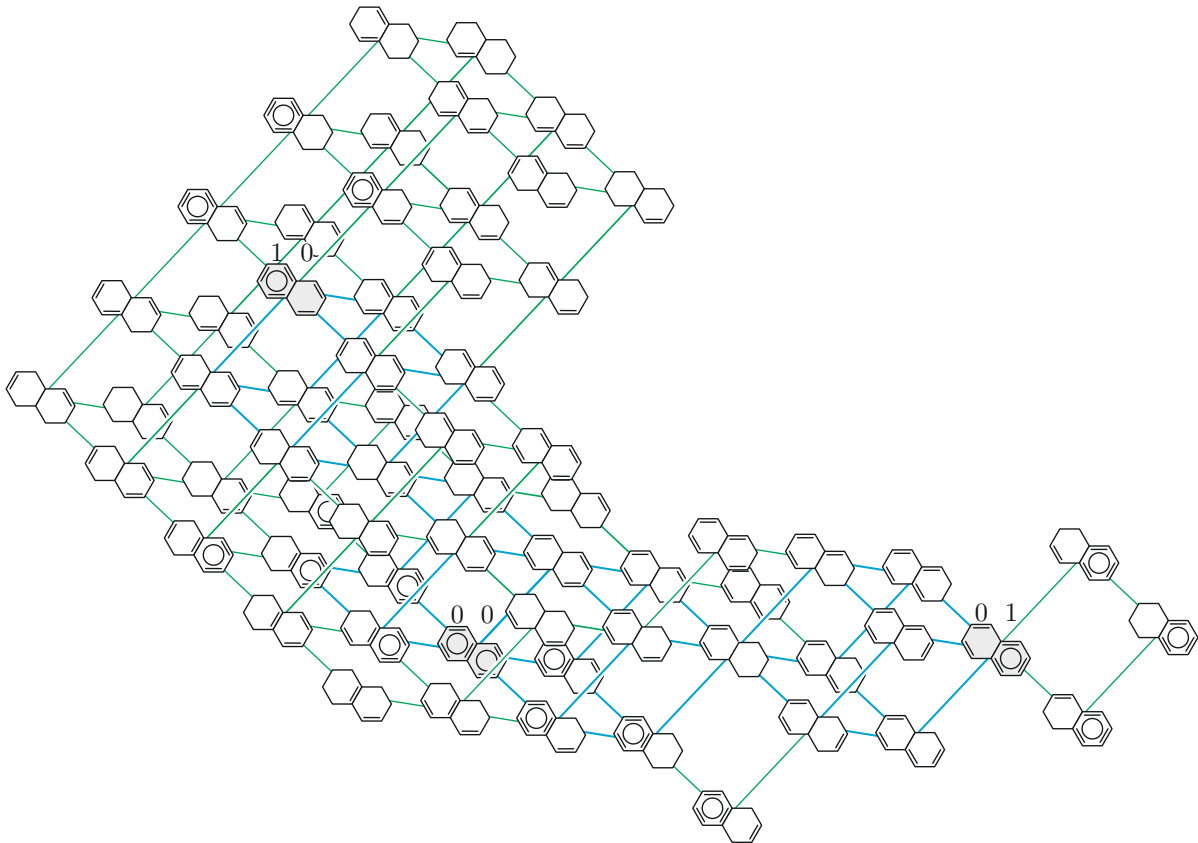


Figure 14. Graph of some matchings if $n = 2$

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