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SEMINORM GENERATING RELATIONS AND THEIR MINKOWSKI FUNCTIONALS

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Abstract. We show that instead of the Minkowski functionals of absorbing, balanced, convex subsets of a vector space X it is more convenient to consider first the Minkowski functionals of balanced valued linear relations of $\mathbb{R}_+$ onto X.

Introduction

A relation F of the set $\mathbb{R}_+$ of all positive numbers onto a vector space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ will be called a seminorm generating relation for X if

(1) $F(r) + F(s) \subset F(r+s)$ for all $r, s \in \mathbb{R}_+$;

(2) $\lambda F(r) \subset F(tr)$ for all $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ with $|\lambda| \leq t$.

This definition is mainly motivated by the fact that if A is an absorbing, balanced, convex subset of X and F_A is a relation on $\mathbb{R}_+$ to X such that

$$F_A(r) = r\,A$$

for all $r \in \mathbb{R}_+$, then F_A is a seminorm generating relation for X.

Moreover, if p is a seminorm on X and F_p and $\bar{F}_p$ are relations on $\mathbb{R}_+$ to X such that

$$F_p(r) = B_r^p(0) \qquad \text{and} \qquad \bar{F}_p(r) = \bar{B}_r^p(0)$$

for all $r \in \mathbb{R}_+$, then F_p and $\bar{F}_p$ are also seminorm generating relations for X.

If F is a seminorm generating relation for X, then the function p_F defined by

$$p_F(x) = \inf\left(F^{-1}(x)\right)$$

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for all $x \in X$, will be called the Minkowski functional of F. Namely, if A is an absorbing, balanced, convex subset of X, then $p_A = p_{F_A}$ is just the usual Minkowski functional of A.

After establishing some easy consequences of the definition of seminorm generating relations, we shall only prove the following basic algebraic properties of the Minkowski functionals.

Theorem 1. *If F is a seminorm generating relation for X, then p_F is a seminorm on X such that $F_{p_F} \subset F \subset \bar{F}_{p_F}$.*

Corollary 1. *If A is an absorbing, balanced, convex subset of X, then p_A is a seminorm on X such that $B_1^{p_A}(0) \subset A \subset \bar{B}_1^{p_A}(0)$.*

Theorem 2. *If p is a seminorm on X and F is a seminorm generating relation for X such that $F_p \subset F \subset \bar{F}_p$, then $p = p_F$.*

Corollary 2. *If p is a seminorm on X and A is an absorbing, balanced, convex subset of X such that $B_1^p(0) \subset A \subset \bar{B}_1^p(0)$, then $p = p_A$.*

Theorem 3. *If F is a seminorm generating relation for X, then p_F is a norm if and only if $\bigcap_{r \in \mathbb{R}_+} F(r) = \{0\}$.*

Corollary 3. *If A is an absorbing, balanced, convex subset of X, then p_A is a norm on X if and only if $\bigcap_{r \in \mathbb{R}_+} rA = \{0\}$.*

Theorem 4. *If F is a seminorm generating relation for X, then $F = F_{p_F}$ if and only if $F(r) = \bigcup_{s<r} F(s)$ for all $r \in \mathbb{R}_+$.*

Corollary 4. *If A is an absorbing, balanced, convex subset of X, then $A = B_1^{p_A}(0)$ if and only if $A = \bigcup_{s<r} sA$.*

Theorem 5. *If F is a seminorm generating relation for X, then $F = \bar{F}_{p_F}$ if and only if $F(r) = \bigcap_{s>r} F(s)$ for all $r \in \mathbb{R}_+$.*

Corollary 5. *If A is an absorbing, balanced, convex subset of X, then $A = \bar{B}_1^{p_A}(0)$ if and only if $A = \bigcap_{s>r} sA$.*

The topological properties of seminorm generating relations and their Minkowski functionals will be investigated elsewhere.

1. Prerequisites

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $X = Y$, then we simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$.

If F is a relation on X to Y, and moreover $x \in X$ and $A \subset X$, then the sets $F(x) = \{ y \in X : \ (x, y) \in F \}$ and $F[A] = \bigcup_{x \in A} F(x)$ are called the images of x and A under F, respectively.

If F is a relation on X to Y, then the sets $D_F = \{x \in X : \ F(x) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the domain and range of F, respectively. If in particular $X = D_F$ (and $Y = R_F$), then we say that F is a relation of X into (onto) Y.

A relation F on X to Y is said to be a function if for each $x \in D_F$ there exists a unique $y \in Y$ such that $y \in F(x)$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If F is a relation on X to Y, then values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse relation F^{-1} of F can be defined such that $F^{-1}(x) = \{ y \in Y : \ x \in F(y) \}$ for all $x \in X$.

Throughout in the sequel, X will denote a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. And for any $\lambda \in \mathbb{K}$ and $A, B \subset X$ we write $\lambda A = \{ \lambda x : \ x \in A \}$ and $A + B = \{ x + y : \ x \in A, \ y \in B \}$.

Note that thus two axioms of a vector space may fail to hold for the family $\mathcal{P}(X)$ of all subsets of X. Namely, only the one-point subsets of X can have additive inverses. Moreover, in general, we only have $(\lambda + \mu) A \subset \lambda A + \mu A$.

If A is a subset of X, then we say that:

(1) A is absorbing if $X = \bigcup_{r \in \mathbb{R}_+} r A$;

(2) A is balanced if $\lambda A \subset A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;

(3) A is convex if $r A + (1 - r) A \subset A$ for all $r \in \mathbb{R}_+$ with $r < 1$.

A function p of X into $\mathbb{R}$ is called a seminorm on X if

$$p(\lambda x) \leq |\lambda| p(x) \qquad \text{and} \qquad p(x + y) \leq p(x) + p(y)$$

for all $\lambda \in \mathbb{K}$ and $x, y \in X$. A seminorm p is called a norm if $p(x) = 0$ implies $x = 0$.

If p is a seminorm on X, then for each $r \in \mathbb{R}_+$ the relations B_r^p and $\bar{B}_r^p$, defined by

$$B_r^p(x) = \left\{ y \in X : \ p(x - y) < r \right\} \quad \text{and} \quad \bar{B}_r^p(x) = \left\{ y \in X : \ p(x - y) \leq r \right\}$$

for all $x \in X$, are called the r-sized open and closed p–surroundings in X, respectively.

Concerning the above basic concepts we shall only need here the following simple theorems.

Theorem 1.1. *If $A \subset X$, then the following assertions hold:*

(1) if A is convex, then $(r + s) A = r A + s A$ for all $r, s \in R_+$;

(2) if A is balanced, then $\lambda A \subset \mu A$ for all $\lambda, \mu \in \mathbb{K}$ with $|\lambda| \leq |\mu|$.

Remark 1.2. Therefore, a balanced subset A of X is absorbing if and only $X = \bigcup_{n=1}^{\infty} n A$.

Theorem 1.3. *If p is a seminorm on X, then*

(1) $p(x) \geq 0$ *for all* $x \in X$;

(2) $p(\lambda x) = |\lambda| p(x)$ *for all* $\lambda \in \mathbb{K}$ *and* $x \in X$.

Remark 1.4. Therefore, our present definition of a seminorm coincides with the usual one.

Theorem 1.5. *If p is a seminorm on X and $r \in \mathbb{R}_+$, then*

(1) $B_r^p(x) = x + B_r^p(0)$ *for all* $x \in X$;

(2) $B_r^p(0)$ *is an absorbing, balanced and convex subset of* X *such that* $B_r^p(0) = r\, B_1^p(0)$.

Remark 1.6. Moreover, the same statements hold for the closed surroundings $\bar{B}_r^p$.

2. Seminorm generating relations

Definition 2.1. A relation F of $\mathbb{R}_+$ onto X will be called a seminorm generating relation for X if

(1) $F(r) + F(s) \subset F(r+s)$ for all $r, s \in \mathbb{R}_+$;

(2) $\lambda F(r) \subset F(tr)$ for all $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ with $|\lambda| \leq t$.

The above definition is mainly motivated by the following simple

Example 2.2. If A is an absorbing, balanced, convex subset of X and F_A is a relation on $\mathbb{R}_+$ to X such that

$$F_A(r) = r\,A$$

for all $r \in \mathbb{R}_+$, then F_A is a seminorm generating relation for X.

Since A is absorbing, for each $x \in X$ there exists an $r \in \mathbb{R}_+$ such that $x \in r\,A$. Hence, it is clear that $A \neq \emptyset$, and thus $\mathbb{R}_+$ is the domain of F_A. Moreover, since $x \in F_A(r)$, it is clear that X is the range of F_A.

On the other hand, if $r, s \in \mathbb{R}_+$, then by Theorem 1.1 (1) it is clear that

$$F_A(r+s) = (r+s)\,A = rA + sA = F_A(r) + F_A(s).$$

Moreover, if $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ such that $|\lambda| \leq t$, then by Theorem 1.1 (2) it is clear that

$$\lambda F_A(r) = \lambda(r\,A) = r(\lambda A) \subset r(t\,A) = (tr)\,A = F_A(tr).$$

Now, as an important particular case of Example 2.2, we can also state

Example 2.3. If p is a seminorm on X and F_p and $\bar{F}_p$ are relations on $\mathbb{R}_+$ to X such that

$$F_p(r) = B_r^p(0) \qquad \text{and} \qquad \bar{F}_p(r) = \bar{B}_r^p(0)$$

for all $r \in \mathbb{R}_+$, then F_p and $\bar{F}_p$ are seminorm generating relations for X.

From Theorem 1.5 (2) we know that $A = B_1^p(0)$ is an absorbing, balanced, convex subset of X such that

$$F_p(r) = B_r^p(0) = r\, B_1^p(0) = r\, A = F_A(r)$$

for all $r \in \mathbb{R}_+$. Therefore, $F_p = F_A$, and thus F_p is a seminorm generating relation for X by Example 2.2.

The fact that $\bar{F}_p$ is also a seminorm generating relation for X can be proved quite similarly by using Remark 1.6 and Example 2.2.

In the sequel, beside Definition 2.1, we shall only need the following obvious

Theorem 2.4. *If F is a seminorm generating relation for X, then*

(1) $0 \in F(r)$ *for all* $r \in \mathbb{R}_+$ *;*

(2) $r\, F(s) \subset F(r\, s)$ *for all* $r, s \in \mathbb{R}_+$ *;*

(3) $F(r) \subset F(s)$ *for all* $r, s \in \mathbb{R}_+$ *with* $r \leq s$ *.*

Proof. Since the assertions (1) and (2) are immediate from the homogenity property 2.1 (2) of F, we need only note that

$$F(r) = F(r) + \{0\} \subset F(r) + F(s - r) \subset F(s)$$

for all $r, s \in \mathbb{R}_+$ with $r < s$. Therefore, the assertion (3) also holds.

However, as a converse to Example 2.2, we can also easily prove the following

Theorem 2.5. *If F is a seminorm generating relation for X, then there exists an absorbing, balanced, convex subset A of X such that $F = F_A$.*

Proof. If $r, s \in \mathbb{R}_+$, then by the homogenity property 2.4 (2) of F we have

$$r\, F(s) \subset F(r\, s).$$

Hence, by writing r^{-1} in place of r, and $r\, s$ in place of s, we can see that

$$r^{-1} F(r\, s) \subset F(s).$$

This implies that $F(r\, s) \subset r\, F(s)$. Therefore, the equality

$$F(r\, s) = r\, F(s)$$

is also true. Hence, under the notation $A = F(1)$, it follows that

$$F(r) = r\, F(1) = r\, A$$

for all $r \in \mathbb{R}_+$.

Threfore, it remains only to prove that A is an absorbing, balanced and convex subset of X . For this, note that if $x \in X$, then since F is onto X there exists an $r \in \mathbb{R}_+$ such that $x \in F(r) = rA$. Therefore, A is absorbing. Moreover, if $\lambda \in \mathbb{K}$ such that $|\lambda| \le 1$, then from the homogenity property 2.1 (2) of F we can at once see that

$$\lambda A = \lambda F(1) \subset F(1) = A .$$

Therefore, A is balanced. Moreover, if $0 < t < 1$, then by the homogenity and additivity properties of F it is clear that

$$t A + (1-t) A = t F(1) + (1-t) F(1) = F(t) + F(1-t) \subset F(1) = A .$$

Therefore, A is convex.

Now, in addition to Theorem 2.4, we can also easily state the following

Theorem 2.6. *If F is a seminorm generating relation for X , then*

(1) $F(rs) = rF(s)$ *for all* $r, s \in \mathbb{R}_+$ *;*

(2) $F(r+s) = F(r) + F(s)$ *for all* $r, s \in \mathbb{R}_+$ *;*

(3) $F(r)$ *is an absorbing, balanced, convex subset of* X *for all* $r \in \mathbb{R}_+$ *.*

Remark 2.7. Note that if F is a balanced valued homogeneous relation of $\mathbb{R}_+$ into X , then

$$\lambda F(r) \subset t F(r) = F(tr)$$

for all $\lambda \in \mathbb{K}$ and $r, t \in \mathbb{R}_+$ with $|\lambda| \le t$. That is, the homogenity property 2.1 (2) also holds.

3. The Minkowski functionals of seminorm generating relations

Definition 3.1. If F is a seminorm generating relation for X , then the function p_F defined by

$$p_F(x) = \inf \left(F^{-1}(x) \right)$$

for all $x \in X$, will be called the Minkowski functional or gauge of F .

Example 3.2. If A is an absorbing, balanced, convex subset of X , then we can at once see that

$$p_{F_A}(x) = \inf \left\{ r \in \mathbb{R}_+ : \quad x \in tA \right\}$$

for all $x \in X$. Therefore, $p_A = p_{F_A}$ is just the usual Minkowski functional of A . (See, [1, p. 24].)

Therefore, it is not surprising that, as a useful reformulation of a well-known theorem on the Minkowski functionals of sets, we have the following

Theorem 3.3. *If F is a seminorm generating relation for X, then p_F is a seminorm on X such that*

$$F_{p_F} \subset F \subset \bar{F}_{p_F}.$$

Proof. If $\lambda \in \mathbb{K}$ and $x \in X$, then by the definition of p_F for each $\varepsilon > 0$ there exists an $r \in F^{-1}(x)$ such that $r < p_F(x) + \varepsilon$. Hence, by noticing that $x \in F(r)$ and using the homogenity property 2.1 (2) of F, we can infer that

$$\lambda x \in \lambda F(r) \subset F(tr),$$

and thus $tr \in F^{-1}(\lambda x)$ for all $t \in \mathbb{R}_+$ with $|\lambda| \leq t$. Hence, since $tr < t\,p_F(x) + t\varepsilon$, it is clear that

$$p_F(\lambda x) = \inf\big(F^{-1}(\lambda x)\big) < t\,p_F(x) + t\varepsilon$$

for all $t \in \mathbb{R}_+$ with $|\lambda| \leq t$. Hence, by letting $t \to |\lambda|$ and $\varepsilon \to 0$, we can infer that

$$p_F(\lambda x) \leq |\lambda|\, p_F(x).$$

On the other hand, if $x, y \in X$, then again by the definition of p_F for each $\varepsilon > 0$ there exist $r \in F^{-1}(x)$ and $s \in F^{-1}(y)$ such that $r < p_F(x) + \varepsilon$ and $s < p_F(y) + \varepsilon$. Hence, by noticing that $x \in F(r)$ and $y \in F(s)$, and using the additivity property 2.1 (1) of F, we can infer that

$$x + y \in F(r) + F(s) \subset F(r+s),$$

and thus $r + s \in F^{-1}(x+y)$. Hence, since $r + s < p_F(x) + p_F(y) + 2\varepsilon$, it is clear that

$$p_F(x+y) = \inf\big(F^{-1}(x+y)\big) < p_F(x) + p_F(y) + 2\varepsilon,$$

and thus

$$p_F(x+y) \leq p_F(x) + p_F(y).$$

Therefore, p_F is a seminorm on X.

Finally, if $r \in \mathbb{R}_+$ and $x \in F_{p_F}(r) = B_r^{p_F}(0)$, i.e., $p_F(x) < r$, then again by the definition of p_F there exists $s \in F^{-1}(x)$ such that $s < r$. Hence, by the monotonicity property 2.4 (3) of F, it is clear that $x \in F(s) \subset F(r)$. Therefore, $F_{p_F}(r) \subset F(r)$.

On the other hand, if $r \in \mathbb{R}_+$ and $x \in F(r)$, then $r \in F^{-1}(x)$. Therefore, by the definition of p_F, we have $p_F(x) \leq r$, and hence $x \in \bar{B}_r^{p_F}(0) = \bar{F}_{p_F}(r)$. Therefore, $F(r) \subset \bar{F}_{p_F}(r)$ is also true.

Now, as an immediate consequence of Example 2.2 and Theorem 3.3, we can also state the following more familiar

Corollary 3.4. *If A is an absorbing, balanced, convex subset of X, then p_A is a seminorm on X such that*
$$B_1^{p_A}(0) \subset A \subset \bar{B}_1^{p_A}(0).$$

In addition, to Theorem 3.3, it is also worth proving the following

Theorem 3.5. *If p is a seminorm on X and F is a seminorm generating relation for X such that*
$$F_p \subset F \subset \bar{F}_p,$$
then $p = p_F$.

Proof. If $x \in X$, then for each $r \in \mathbb{R}_+$, with $p(x) < r$, we have
$$x \in B_r^p(0) = F_p(r) \subset F(r).$$
Therefore, $r \in F^{-1}(x)$, and thus
$$p_F(x) = \inf\big(F^{-1}(x)\big) \leq r.$$
Hence, by letting $r \to p(x)$, we can infer that $p_F(x) \leq p(x)$.

On the other hand, by the definition of $p_F(x)$, for each $\varepsilon > 0$ there exists an $r \in F^{-1}(x)$ such that $r < p_F(x) + \varepsilon$. Hence, we can see that
$$x \in F(r) \subset \bar{F}_p(r) = \bar{B}_r^p(0).$$
Therefore, $p(x) \leq r < p_F(x) + \varepsilon$, and thus $p(x) \leq p_F(x)$ is also true.

Remark 3.6. In particular, by Theorem 3.5, we have $p = p_{F_p} = p_{\bar{F}_p}$ for every seminorm p on X.

Moreover, as an immediate consequence of Example 2.3 and Theorem 3.5, we can also state

Corollary 3.7. *If p is a seminorm on X and A is an absorbing, balanced, convex subset of X such that*
$$B_1^p(0) \subset A \subset \bar{B}_1^p(0),$$
then $p = p_A$.

4. Some further properties of the Minkowski functionals

Theorem 4.1. *If F is a seminorm generating relation for X, then the following assertions are equivalent:*

(1) p_F *is a norm;* *(2)* $\displaystyle\bigcap_{r \in \mathbb{R}_+} F(r) = \{0\}$.

Proof. If $x \in F(r)$, and hence $r \in F^{-1}(x)$ for all $r \in \mathbb{R}_+$, then by the definition of p_F we have $p_F(x) = 0$. Hence, if the assertion (1) holds, it follows that $x = 0$. Therefore, since $0 \in F(r)$ for all $r \in \mathbb{R}_+$, the assertion (2) also holds.

While, if $x \in X$ such that $p_F(x) = 0$, then by the definition of p_F for each $r \in \mathbb{R}_+$ there exists $s \in F^{-1}(x)$ such that $s < r$. Hence, by the monotonocity property of F, it is clear that $x \in F(s) \subset F(r)$. Therefore, if the assertion (2) holds, then $x = 0$, and thus the assertion (1) also holds.

Corollary 4.2. *If A is an absorbing, balanced, convex subset of X, then the following assertions are equivalent:*

(1) $p_{_A}$ *is a norm;* *(2)* $\bigcap_{r\in\mathbb{R}_+} r\,A = \{\,0\,\}$.

Remark 4.3. Note that, since A is balanced, we may write $\mathbb{K}\setminus\{0\}$ in place of $\mathbb{R}_+$ in the assertion (2).

Theorem 4.4. *If F is a seminorm generating relation for X, then the following assertions are equivalent:*

(1) $F = F_{p_{_F}}$; *(2)* $F(r) = \bigcup_{s<r} F(s)$ *for all* $r\in\mathbb{R}_+$.

Proof. If $r\in\mathbb{R}_+$ and $x\in F(r)$, and the assertion (1) holds, then we have $x\in F_{p_{_F}}(r) = B_r^{\,p_F}(0)$. Hence, it follows that

$$\inf\big(F^{-1}(x)\big) = p_{_F}(x) \,<\, r\,.$$

Therefore, there exists an $s\in F^{-1}(x)$ such that $s<r$. Hence, it follows that $x\in F(s)$. Therefore,

$$F(r)\,\subset\,\bigcup_{s<r} F(s)\,.$$

Now, since the converse inlusion is immediate from the monotonicity property of F, it is clear that the assertion (2) also holds.

While, if $r\in\mathbb{R}_+$ and $x\in F(x)$, and the assertion (2) holds, then there exists an $s<r$ such that $x\in F(s)$, and hence $s\in F^{-1}(x)$. Therefore,

$$p_{_F}(x) = \inf\big(F^{-1}(x)\big) \le s \,<\, r\,,$$

and hence $x\in B_r^{\,p_F}(0) = F_{p_{_F}}(r)$. Consequently, we have $F(r)\subset F_{p_{_F}}(r)$. Now, since the converse inclusion is always true by Theorem 3.3, it is clear that the assertion (1) also holds.

Corollary 4.5. *If A is an absorbing, balanced, convex subset of X, then the following assertions are equivalent:*

(1) $A = B_1^{\,p_A}(0)$; *(2)* $A = \bigcup_{s<1} s\,A$.

Theorem 4.6. *If F is a seminorm generating relation for X, then the following assertions are equivalent:*

(1) $F = \bar{F}_{p_{_F}}$; *(2)* $F(r) = \bigcap_{s>r} F(s)$ *for all* $r\in\mathbb{R}_+$.

Proof. If $r\in\mathbb{R}_+$, and $x\in X$ such that $x\in F(s)$, i.e., $s\in F^{-1}(x)$ for all $s>r$, then

$$p_{_F}(x) = \inf\big(F^{-1}(x)\big) \,\le\, r\,,$$

and hence $x \in \bar{B}_r^{p_F}(0) = \bar{F}_{p_F}(r)$. Therefore, if the assertion (1) holds, then we also have $x \in F(r)$. Consequently,

$$\bigcap_{s>r} F(s) \subset F(r).$$

Hence, since the converse inclusion is immediate from the monotonicity property of F, it is clear that the assertion (2) also holds.

While, if $r \in \mathbb{R}_+$, and $x \in \bar{F}_{p_F}(r)$, i.e., $x \in \bar{B}_r^{p_F}(0)$, then

$$\inf\big(F^{-1}(x)\big) = p_F(x) \leq r.$$

Therefore, for each $s > r$ there exists a $t \in F^{-1}(x)$ such that $t < s$. Hence, by the monotonicity property of F, it is clear that $x \in F(t) \subset F(s)$. Therefore,

$$\bar{F}_{p_F}(r) \subset \bigcap_{s>r} F(s).$$

Hence, if the assertion (2) holds, it follows that $\bar{F}_{p_F}(r) \subset F(r)$. Now, since the converse inclusion is always true by Theorem 3.3, it is clear that the assertion (1) also holds.

Corollary 4.7. *If A is an absorbing, balanced, convex subset of X, then the following assertions are equivalent:*

(1) $A = \bar{B}_1^{p_A}(0)$; *(2)* $A = \bigcap_{s>1} sA$.

Remark 4.8. The above corollaries are not established in the standard books on functional analysis.

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