

INTEGRAL REPRESENTATION OF BOUNDED AND ABSOLUTELY INTEGRABLE FUNCTIONS

KAMEL AL-KHALED

ABSTRACT. In this paper, we obtain an integral representation formula for an even function, as a consequence, we show that if the function satisfying some conditions over $(0, 1)$ then it is completely characterized by its value in the neighborhood of 1.

1. INTRODUCTION

Let $f(x)$ be an even function over $-1 \leq x \leq 1$ and $G(u)$ is any even bounded function and integrable over the interval $-1 \leq u \leq 1$. In the first theorem we will show that the function $f(x)$ can be written as an integral representation of the function $G(u)$. Then we proved that if $f(x)$ is bounded and absolutely integrable over the interval $(0, 1 - \epsilon)$, and satisfy the integral representation of $f(x)$ is bounded and absolutely integrable over the interval $(0, 1)$.

Integrable functions have frequently appeared in the literature of the last few years, for example, see [1], [3] and [4]. Before proving the main result we state and proof the following theorem.

Theorem 1.1. *Suppose $f(x)$ is even function over $-1 \leq x \leq 1$, and $G(u)$ be an even bounded integrable function over the interval $-1 \leq u \leq 1$. And that $G(u)$ together with its derivatives of all orders is continuous over the interval $(-1, 1)$ and that it vanishes with all its derivatives for $u = \mp 1$. Then, for $\int_{-1}^1 G(u)du \neq 0$, we have*

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{n! \int_{-1}^1 G(u)du} \int_{-1}^1 \frac{d^{n+1}}{du^{n+1}} \left[(u-x)^n \int_{-1}^u G(\bar{u})d\bar{u} \right] f(u)du$$

Proof. Taylor series for $f(x)$ is given by

$$f(x) = f(u) + (x-u)f'(u) + \cdots + \frac{(x-u)^n}{n!} f^{(n)}(u) + \cdots$$

which we shall suppose uniformly convergent in the real argument u for $-1 \leq u \leq 1$ and for every x such that $-1 \leq x \leq 1$. Since $G(u)$ is bounded and integrable over the interval

1991 *Mathematics Subject Classification.* 35C20, 31B10.

Key words and phrases. integral representation, integrable functions, expansion.

$-1 \leq u \leq 1$. Then

$$(1.1) \quad \begin{aligned} f(x) \int_{-1}^1 G(u) du &= \int_{-1}^1 f(u) G(u) du + \int_{-1}^1 (x-u) f'(u) G(u) du + \cdots \\ &\quad + \frac{1}{n!} \int_{-1}^1 (x-u)^n f^{(n)}(u) G(u) du \end{aligned}$$

with the assumption that $G(u)$ together with its derivatives of all orders is continuous over the interval $(-1, 1)$, and that it vanishes with all its derivatives for $u = \mp 1$. Then, integrating by parts yields

$$(1.2) \quad \begin{aligned} \int_{-1}^1 (x-u)^n f^{(n)}(u) G(u) du &= \int_{-1}^1 (x-u)^n G(u) d\{f^{(n-1)}(u)\} \\ &= \left[(x-u)^n G(u) f^{(n-1)}(u) \right]_{-1}^1 - \int_{-1}^1 f^{(n-1)}(u) d\{(x-u)^n G(u)\} \\ &= - \int_{-1}^1 \frac{d}{du} \{(x-u)^n G(u)\} f^{(n-1)}(u) du = - \int_{-1}^1 \frac{d}{du} \{(x-u)^n G(u)\} d\{f^{(n-2)}(u)\} \\ &= - \left\{ \frac{d}{du} [(x-u)^n G(u)] f^{(n-2)}(u) \right\}_{-1}^1 + \int_{-1}^1 f^{(n-2)}(u) \left\{ \frac{d^2}{du^2} [(x-u)^n G(u)] \right\} du \\ &\vdots \\ &= \int_{-1}^1 \frac{d^n}{du^n} [(u-x)^n G(u)] f(u) du \end{aligned}$$

We shall now use the fact that $\int_{-1}^1 G(u) du \neq 0$. We consequently obtain from (1.1) and (1.2) the formula

$$(1.3) \quad \begin{aligned} f(x) &= \frac{1}{\int_{-1}^1 G(u) du} \left\{ \int_{-1}^1 G(u) f(u) du + \int_{-1}^1 \frac{d}{du} [(u-x)G(u)] f(u) du + \cdots \right. \\ &\quad \left. + \frac{1}{n!} \int_{-1}^1 \frac{d^n}{du^n} [(u-x)^n G(u)] f(u) du + \cdots \right\} \end{aligned}$$

Set $\int_{-1}^u G(\tilde{u}) d\tilde{u} = F(u)$. Then $F(u)$ will then be characterized by the same properties as those have determined for $G(u)$, as to the existence and continuity of its derivatives, and as to the vanishing of the function and its derivatives at the end of the interval, except that $F(1) \neq 0$. Let us consider the expression

$$\frac{1}{n!} \frac{d^{n+1}}{du^{n+1}} [(u-x)^n F(u)]$$

Clearly,

$$\begin{aligned} \frac{1}{n!} \frac{d^{n+1}}{du^{n+1}} [(u-x)^n F(u)] &= \frac{1}{n!} \frac{d^n}{du^n} \frac{d}{du} [(u-x)^n F(u)] \\ &= \frac{1}{n!} \frac{d^n}{du^n} [n(u-x)^{n-1} F(u) + (u-x)^n G(u)] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(n-1)!} \frac{d^n}{du^n} \left[(u-x)^{n-1} F(u) \right] + \frac{1}{n!} \frac{d^n}{du^n} \left[(x-u)^n G(u) \right] \\
 &= \frac{1}{(n-2)!} \frac{d^{n-1}}{du^{n-1}} \left[(u-x)^{n-2} F(u) \right] \\
 &\quad + \frac{1}{(n-1)!} \frac{d^{n-1}}{du^{n-1}} \left[(x-u)^{n-1} F(u) \right] + \frac{1}{n!} \frac{d^n}{du^n} \left[(u-x)^n G(u) \right] \\
 &= G(u) + \frac{d}{du} \left[(u-x)G(u) \right] + \cdots + \frac{1}{n!} \frac{d^n}{du^n} \left[(u-x)^n G(u) \right]
 \end{aligned}$$

Hence (1.3) becomes

$$(1.4) \quad f(x) = \lim_{n \rightarrow \infty} \frac{1}{n! F(1)} \int_{-1}^1 \frac{d^{n+1}}{du^{n+1}} \left[(u-x)^n F(u) \right] f(u) du$$

But $G(x)$ and $f(x)$ are even functions, so (1.4) becomes

$$(1.5) \quad f(x) = \lim_{n \rightarrow \infty} \frac{1}{n! F(1)} \int_0^1 \frac{d^{n+1}}{du^{n+1}} \left\{ \left[(u-x)^n + (u+x)^n \right] F(u) \right\} f(u) du$$

which ends the proof of the theorem. $\square$

2. THE MAIN RESULT

We will show that if we take our $G(u)$ the function $\exp(1/(u^2-1))$, the difference between (1.5) and an expression of the Fourier type is really essential. In particular, we will show that if $f(x)$ is bounded and absolutely integrable over $(0, 1-\epsilon)$, is zero over $(1-\epsilon, 1)$, and satisfies (1.5) at every point of $(0, 1)$, then it is identically zero over $(0, 1)$. From this it will follow at once that a function satisfying (1.5) over $(0, 1)$, bounded, and absolutely integrable, is completely characterized by its value in the neighborhood of 1. It is not even necessary, however, that the function satisfy (1.5) over the whole of $(0, 1)$; it is a sufficient condition that the following limit exist

$$\lim_{n \rightarrow \infty} \frac{2}{n! F(1)} \int_0^1 \frac{d^{n+1}}{du^{n+1}} \left[u^n F(u) \right] f(u) du$$

Define the auxiliary function of a complex variable by

$$\phi(\xi) = \frac{2\xi}{\int_{-1}^1 \exp(1/(x^2-1)) dx} \int_0^{1-\epsilon} \exp(1/(\xi^2 u^2 - 1)) f(u) du$$

To find the singularities of the function $\phi(\xi)$ note that $|\exp(1/(\xi^2 u^2 - 1))| \leq |1/(\xi^2 u^2 - 1)|$. If $|\xi u + 1| > \eta$, $|\xi u - 1| > \eta$, we have $|\exp(1/(\xi^2 u^2 - 1))| < \exp(1/\eta^2)$. Now define the region Γ in the complex plane in which ξ lies when $|\xi u + 1| > \eta$, $|\xi u - 1| > \eta$ for every u in the interval $(0, 1-\epsilon)$. In the region Γ , the function $\exp(1/(\xi^2 u^2 - 1)) f(u)$ is uniformly bounded and integrable in u , so that $\phi(\xi)$ is defined. The related function

$$\begin{aligned}
 \phi'(\xi) &= \frac{2}{\int_{-1}^1 \exp(\frac{1}{x^2-1}) dx} \int_0^{1-\epsilon} \exp\left(\frac{1}{\xi^2 u^2 - 1}\right) f(u) du - \\
 &\quad - \frac{4\xi^2}{\int_{-1}^1 \exp(\frac{1}{x^2-1}) dx} \int_0^{1-\epsilon} \frac{u^2}{(\xi^2 u^2 - 1)^2} \exp\left(\frac{1}{\xi^2 u^2 - 1}\right) f(u) du
 \end{aligned}$$

may be proved to exist by a similar argument over the same region Γ . There is no difficulty in showing directly that

$$\lim_{|\lambda| \rightarrow 0} \frac{\phi(\xi + \lambda) - \phi(\xi)}{\lambda} = \phi'(\xi)$$

whenever ξ and $\xi + \lambda$ lie in the region Γ . Hence ϕ is analytic over Γ . Now let us consider $\phi(x/(1-y))$ as a function of y , given that $|x| < 1/(1-\epsilon)$. It is clearly that ϕ is analytic in a neighborhood containing the origin, as is also $1/(1-y)\phi(x/(1-y))$. Let us put $\int_0^z \phi(\bar{z})d\bar{z} = \Phi(z)$, where the path of integration lies entirely within the circle of convergence of the Taylor series about the origin for $\phi(x)$. We shall then have

$$\begin{aligned} (2.1) \quad \frac{1}{1-y}\phi\left(\frac{x}{1-y}\right) &= \frac{\partial}{\partial x}\Phi\left(\frac{x}{1-y}\right) \\ &= \left[\frac{\partial}{\partial x}\Phi\left(\frac{x}{1-y}\right)\right]_{y=0} + y\left[\frac{\partial^2}{\partial x\partial y}\Phi\left(\frac{x}{1-y}\right)\right]_{y=0} + \cdots \\ &\quad + \frac{y^n}{n!}\left[\frac{\partial^{n+1}}{\partial x\partial y^n}\Phi\left(\frac{x}{1-y}\right)\right]_{y=0} + \cdots \end{aligned}$$

Now let $x/(1-y) = z$, or $z = x + yz$. Then

$$\frac{\partial z}{\partial y} = \frac{x}{(1-y)^2} = z \frac{\partial z}{\partial x}$$

Hence,

$$\frac{\partial \Phi(z)}{\partial y} = \Phi'(z) \frac{\partial z}{\partial y} = z \Phi'(z) \frac{\partial z}{\partial x} = z \frac{\partial \Phi(z)}{\partial x}$$

Again,

$$\begin{aligned} \frac{\partial^2 \Phi(z)}{\partial y^2} &= \frac{\partial}{\partial y} \left(z \frac{\partial \Phi(z)}{\partial x} \right) = \frac{\partial z}{\partial y} \frac{\partial \Phi(z)}{\partial x} + z \frac{\partial^2 \Phi(z)}{\partial x \partial y} \\ &= \Phi'(z) \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + z \frac{\partial^2 \Phi(z)}{\partial x \partial y} = \frac{\partial \Phi(z)}{\partial y} \frac{\partial z}{\partial x} + z \frac{\partial^2 \Phi(z)}{\partial x \partial y} = \frac{\partial}{\partial x} \left(z^2 \frac{\partial \Phi(z)}{\partial x} \right) \end{aligned}$$

In general,

$$\frac{\partial^n \Phi(z)}{\partial y^n} = \frac{\partial^{n-1}}{\partial x^{n-1}} \left(z^n \frac{\partial \Phi(z)}{\partial x} \right)$$

Hence,

$$\left[\frac{\partial^{n+1} \Phi(z)}{\partial x \partial y^n} \right]_{y=0} = \left[\frac{\partial^n}{\partial x^n} \left(z^n \frac{\partial \Phi(z)}{\partial x} \right) \right]_{y=0} = \frac{\partial^n}{\partial x^n} [x^n \phi(x)]$$

Formula (2.1) thus becomes

$$(2.2) \quad \frac{1}{1-y}\phi\left(\frac{x}{1-y}\right) = \phi(x) + y \frac{d}{dx}(x\phi(x)) + \cdots + \frac{y^n}{n!} \frac{d^n}{dx^n}(x^n \phi(x)) + \cdots$$

It has been given here for the purpose of showing that there is actually a region for which the two sides of (2.2) are identical, provided that as in the present case the radius of convergence of the MacLaurin series for $\phi(x)$ is greater than 1.

We now say that if $\lim_{n \rightarrow \infty} \frac{1}{n!} \left[\frac{d^n}{dx^n} (x^n \phi(x)) \right]_{x=1}$ exists, $\phi(x)$ is identically zero. To establish this, a consideration of the singularities of ϕ is sufficient. To begin with, ϕ is an odd function, and its singularities always occur in pairs. Again, we have already seen that all

the singularities of ϕ lie on the real axis, with a modulus greater than $1/(1-\epsilon)$. Now, since we may write (2.2) in the form

$$(2.3) \quad \begin{aligned} \phi\left(\frac{1}{1-y}\right) &= \phi(1) + y \left\{ \left[\frac{d}{dx}(x\phi(x)) \right]_{x=1} - \phi(1) \right\} + \cdots \\ &+ y^n \left\{ \frac{1}{n!} \left[\frac{d^n}{dx^n}(x^n\phi(x)) \right]_{x=1} - \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\phi(x)) \right]_{x=1} \right\} + \cdots \end{aligned}$$

and since this power series converges for $y = 1$, it follows that ϕ has no singularities on the finite positive real axis, and hence no singularities on the real axis at all, except possibly at infinity. The singularities at infinity, for a function with only one singularity must be single-valued (see, [3]).

Let $y \rightarrow 1$ along any path for which $\arg(1/(1-y))$ lies between $-\sin^{-1}\eta$ and $\sin^{-1}\eta$. Since the power series (2.3) converges to $\lim_{n \rightarrow \infty} \frac{1}{n!} \left[\frac{d^n}{dx^n}(x^n\phi(x)) \right]_{x=1}$, if this quantity exists, it follows that

$$\lim_{y \rightarrow 1} \phi\left(\frac{1}{1-y}\right) = \lim_{n \rightarrow \infty} \frac{1}{n!} \left[\frac{d^n}{dx^n}(x^n\phi(x)) \right]_{x=1}$$

The $\lim_{x \rightarrow 1} \phi(1/(1-y))$ will also exist if $y \rightarrow 1$ for any path for which $\arg(1/(1-y))$ lies between $\pi - \sin^{-1}\eta$ and $\pi + \sin^{-1}\eta$, since ϕ is odd.

Now consider the function $\phi(\xi)/\xi$, this has no singularities at the origin, and is uniformly bounded whenever $\arg(\xi)$ lies outside of the angles $(-\sin^{-1}\eta, \sin^{-1}\eta)$ and $(\pi - \sin^{-1}\eta, \pi + \sin^{-1}\eta)$. All this follows from the uniformly bounded and integrable character of $\exp(1/(\xi^2 u^2 - 1))f(u)$. On the other hand, it follows from what we have just seen that if $\xi \rightarrow \infty$ along any path within the angles $(-\sin^{-1}\eta, \sin^{-1}\eta)$ and $(\pi - \sin^{-1}\eta, \pi + \sin^{-1}\eta)$, then $\lim_{\xi \rightarrow \infty} \phi(\xi)/\xi = 0$. It follows that $\phi(\xi)/\xi$ can neither have a pole nor an essential singularity anywhere, and so reduce to a constant, which can only be zero. Hence $\phi(\xi) \equiv 0$.

Now let $f(u) = \sum_{m=0}^{\infty} a_m u^m$. Then $G(u) = \sum_{m=0}^{\infty} m a_m u^{m-1}$ and

$$\begin{aligned} \phi(\xi) &= \frac{2\xi}{F(1)} \int_0^{1-\epsilon} \left\{ \sum_{m=0}^{\infty} m a_m (\xi u)^{m-1} \right\} f(u) du \\ &= \sum_{m=0}^{\infty} \frac{2\xi^m}{F(1)} \int_0^{1-\epsilon} m a_m u^{m-1} f(u) du, |\xi| < \frac{1}{1-\epsilon} \end{aligned}$$

and so,

$$\begin{aligned} \frac{1}{n!} \left[\frac{d^n}{dx^n}(x^n\phi(x)) \right]_{x=1} &= \\ &= \left\{ \sum_{m=0}^{\infty} \frac{2(m+n)(m+n-1)\cdots(m+1)m}{n!F(1)} x^m \int_0^{1-\epsilon} a_m u^{m-1} f(u) du \right\}_{x=1} \\ &= \frac{2}{n!F(1)} \int_0^{1-\epsilon} \frac{d^{n+1}}{du^{n+1}} [u^n F(u)] f(u) du. \end{aligned}$$

That is the validity of (1.5) for $x = 0$ involves the identical vanishing of $\phi(\xi)$. In other words, if (1.5) holds,

$$(2.4) \quad \int_0^{1-\epsilon} \exp\left(\frac{1}{\xi^2 u^2 - 1}\right) f(u) du = 0, \forall \xi.$$

Let us now consider the sequence of derivatives of $\exp(1/(\xi^2 u^2 - 1))f(u)$. and note that the derivative is of the form

$$\left[\frac{2A_1}{x-1} + \frac{2A_2}{(x-1)^3} + \cdots + \frac{2A_{2n-1}}{(x-1)^{2n-1}} \right] \exp\left(\frac{1}{x-1}\right)$$

where the A' s are positive or negative integers. If we differentiate this expression we get

$$\left[\frac{-2A_1}{(x-1)^2} - \frac{4A_2}{(x-1)^3} - \cdots \mp \frac{2n}{(x-1)^{2n+1}} - \frac{2A_1}{(x-1)^3} - \cdots - \frac{2A_{2n-1}}{(x-1)^{2n-1}} \mp \frac{1}{(x-1)^{2n+2}} \right] \exp\left(\frac{1}{x-1}\right)$$

which is of the same form. Hence by mathematical induction, every derivative of $\exp(1/(1-x))$ is of this form. It follows that there is an integer k such that

$$\left[\frac{d^n}{dx^n} \exp\left(\frac{1}{x-1}\right) \right]_{x=0} = \frac{2k+1}{e} \neq 0$$

so that

$$(2.5) \quad \left[\frac{d^{2n}}{dx^{2n}} \exp\left(\frac{1}{x^2-1}\right) \right]_{x=0} \neq 0$$

as is obvious from a comparison of the Taylor series for $\exp(1/(1-x))$ and $\exp(1/(x^2-1))$. It follows from (2.4) on differentiation that

$$0 = \int_0^{1-\epsilon} \left[\frac{\partial^{2n}}{\partial \xi^{2n}} \exp\left(\frac{1}{\xi^2 u^2 - 1}\right) \right]_{\xi=0} f(u) du = \int_0^{1-\epsilon} u^{2n} \left[\frac{d^{2n}}{dx^{2n}} \exp\left(\frac{1}{x^2-1}\right) \right]_{x=0} f(u) du$$

Hence by (2.5)

$$\int_0^{1-\epsilon} u^{2n} f(u) du = 0, \quad \forall n.$$

it is a direct consequence from this and the fact that the even powers of u forms a complete set over the interval $(0, 1-\epsilon)$ (as follows from Weierstrass's Theorem on polynomials representation) that except for a set of points of zero measure, $f(u) = 0$ over $(0, 1-\epsilon)$. From this and (1.5) it follows again that $f(u) = 0$ everywhere over $(0, 1)$. This complete the proof of our Theorem.

REFERENCES

- [1] A. Chakrabarti and A. J. George. Diagonalizable generalized Abel integral operators. *SIAM J. Appl. Math.* 57, 2:568–575, 1997.
- [2] Duffy, Dean G. *Transform methods for solving partial differential equations*. CRC, 1994.
- [3] A. R. Forsyth. *Theory of functions of a complex variable*. New York, Dover, 1965.
- [4] J. Choi and C. Nash. *Integral representation of the Kinkelin's constant* *A. Math. Japanica*, 45, 2:223–230, 1997.
- [5] M. Medved. A new approach to an analytic of Henry type integral inequalities and their bihari type versions. *J. Math. Anal. Appl.* 214, 349–366, 1997.
- [6] D. C. Mitrinovic and J. K. Keckic. *The Cauchy method of residues*. Boston, 1994.
- [7] E. C. Titchmarsh. *Introduction to the theory of Fourier integrals*. Oxford, 1948.

Received September 14, 1999.

DEPARTMENT OF APPLIED MATHEMATICAL SCIENCES
JORDAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
IRBID 22110, JORDAN
E-mail address: applied@just.edu.jo