

ON WEAKLY SYMMETRIC AND WEAKLY RICCI-SYMMETRIC K -CONTACT MANIFOLDS

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1. INTRODUCTION

The notion of locally symmetric Riemannian manifolds has been weakened in several ways and to a different extent by M. C. CHAKI [2], [3], M. C. CHAKI and S. K. SAHA [4], L. TAMÁSSY and T. Q. BINH [9], introducing the notions of pseudosymmetric, pseudo Ricci-symmetric, weakly symmetric and weakly projective symmetric Riemannian manifolds. Manifolds of such kinds were investigated by the above authors in the above mentioned papers and also in [5], [6]. On the other hand, DEBASIS TARAFDAR and U. C. DE [8] revealed the incompatibility of K -contact structure with pseudosymmetry and pseudo Ricci-symmetry, provided these notions do not reduce to simple symmetry.

In this paper we shall give necessary conditions for the compatibility of several K -contact structures with weak symmetry and weak Ricci-symmetry and weak Ricci-symmetry, provided they do not reduce to the common local symmetry, that is, they are proper. Thus weak symmetry and weak Ricci-symmetry are weaker than pseudosymmetry and pseudo Ricci-symmetry respectively.

In a recent paper [10] L. TAMÁSSY and T. Q. BINH studied weakly symmetric and weakly Ricci-symmetric Sasakian manifolds. It is known that every Sasakian manifold is K -contact, but the converse is not true in general. However, a 3-dimensional K -contact manifold is Sasakian. This enables us to get back TAMÁSSY and BINH's result [10] from our theorems.

2. WEAKLY SYMMETRIC AND WEAKLY RICCI-SYMMETRIC MANIFOLDS

The notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. TAMÁSSY and T. Q. BINH [9], [10].

A non-flat Riemannian manifold (M^n, g) ($n > 2$) is called *weakly symmetric* if there exist 1-forms $\alpha, \beta, \gamma, \delta$ and σ such that

$$(2.1) \quad (\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) + \beta(Y)R((X, Z, U, V) \\ + \gamma(Z)R(Y, X, U, V) + \delta(U)R(Y, X, V) \\ + \sigma(V)R(Y, Z, U, X)$$

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holds for all vector fields $X, Y, \dots, V \in \mathfrak{X}(M)$, where R is the Riemannian curvature tensor of (M^n, g) of type $(0, 4)$ and ∇ is the covariant differentiation with respect to the Riemannian metric g . A weakly symmetric manifold is said to be *proper* if $\alpha = \beta = \gamma = \delta = \sigma = 0$ is not the case.

A Riemannian manifold (M^n, g) ($n > 2$) is called *weakly Ricci-symmetric* if there exist 1-forms ρ, μ, ν such that the relation

$$(2.2) \quad (V_X S)(Y, Z) = \rho(X)S(Y, Z) + \mu(Y)S(X, Z) + \nu(Z)S(X, Y)$$

holds for any vector fields X, Y, Z where S is the Ricci tensor of type $(0, 2)$ of the manifold M^n . A weakly Ricci-symmetric manifold is said to be *proper* if $\rho = \nu = \mu = 0$ is not the case.

Recently U. C. DE and S. BANDYOPADHYAY [6] gave an example of a weakly symmetric manifold and found its reduced form as follows:

$$(2.3) \quad (\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) \\ + \beta(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V) \\ + \sigma(V)R(Y, Z, U, X).$$

Let $\{e_i\}$, ($i = 1, 2, \dots, n$) be an orthonormal basis of the tangent space at point of the manifold. Then, putting $Y = V = e_i$ in (2.3) and taking summation for $1 \leq i \leq n$, we obtain

$$(2.4) \quad (V_X S)(Z, U) = \alpha(X)S(Z, U) + \beta(Z)S(X, U) + \delta(U)S(Z, X) \\ + \beta(R(X, Z)U) + \delta(R(X, U)Z).$$

3. K -CONTACT RIEMANNIAN MANIFOLDS

Let (M^n, g) be a contact Riemannian manifold with contact form η , associated vector field ξ , $(1, 1)$ -tensor field φ and associated Riemannian metric g . If ξ is a Killing vector field, then (M^n, g) is called a K -contact Riemannian manifold [1], [7]. A K -contact manifold is Sasakian if and only if the relation

$$(3.1) \quad (\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X \quad \text{holds for all } X, Y.$$

In a contact Riemannian manifold (M^n, g) the following relations hold [1], [7]:

$$(3.2) \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta\varphi = 0,$$

$$(3.3) \quad \varphi^2 X = -X + \eta(X)\xi,$$

$$(3.4) \quad g(X, \xi) = \eta(X),$$

$$(3.5) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y .

If (M^n, g) is a K -contact manifold, then besides (3.2)–(3.5), the following relations hold [1], [7]:

$$(3.6) \quad \nabla_X \xi = -\varphi X,$$

$$(3.7) \quad g(R(\xi, X)Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(3.8) \quad S(X, \xi) = (n-1)\eta(X),$$

$$(3.9) \quad R(\xi, X)\xi = -X + \eta(X)\xi,$$

$$(3.10) \quad (\nabla_X \varphi)(Y) = R(\xi, X)Y \quad \text{for any vector fields } X, Y.$$

Further, since ξ is a Killing vector, the Lie derivative of S and the scalar curvature r vanish, i.e.

$$(3.11) \quad L_\xi S = 0$$

and

$$(3.12) \quad L_\xi r = 0.$$

4. WEAKLY SYMMETRIC K -CONTACT MANIFOLDS

We suppose that the weakly symmetric manifold (M^n, g) ($n > 3$) is K -contact. Since the manifold is weakly symmetric, we have (2.4) which, by putting $X = \xi$ and using (3.8), yields

$$(4.1) \quad (\nabla_\xi S)(Z, U) = \alpha(\xi)S(Z, U) + (n-1)[\beta(Z)\eta(U) + \delta(U)\eta(Z)] \\ + \beta(R(\xi, Z)U) + \delta(R(\xi, U)Z).$$

From (3.11), it follows that

$$(4.2) \quad (\nabla_\xi S)(Z, U) = -S(\nabla_Z \xi, U) - S(Z, \nabla_U \xi).$$

By virtue of (3.6), we get from (4.2)

$$(4.3) \quad (\nabla_\xi S)(Z, U) = S(\varphi Z, U) + S(Z, \varphi U).$$

Now, since φ is skew-symmetric, the Ricci operator Q is symmetric and $Q\varphi = \varphi Q$ in a K -contact manifold, we obtain from (4.3)

$$(4.4) \quad (\nabla_\xi S)(Z, U) = 0,$$

where the Ricci operator Q is associated with S by $g(QX, Y) = S(X, Y)$.

From (4.1) and (4.4), we have

$$(4.5) \quad \alpha(\xi)S(Z, U) + (n-1)[\beta(Z)\eta(U) + \delta(U)\eta(Z)] \\ + \beta(R(\xi, Z)U) + \delta(R(\xi, U)Z) = 0.$$

Putting $Z = U = \xi$ in (4.5) and then using (3.8) and (3.2), we get

$$(n-1)[\alpha(\xi) + \beta(\xi) + \delta(\xi)] = 0,$$

which gives us (since $n > 3$),

$$(4.6) \quad \alpha(\xi) + \beta(\xi) + \delta(\xi) = 0.$$

This means that the vanishing of the 1-form $\alpha + \beta + \delta$ over the Killing vector field ξ of (M^n, g) ($n > 3$) is necessary in order that the manifold (M^n, g) ($n > 3$) be a K -contact manifold. We show that $\alpha + \beta + \delta = 0$ is also necessary for this.

Now, putting $U = \xi$ in (2.4) and then using (3.8), we get

$$(4.7) \quad (\nabla_X S)(Z, \xi) = (n-1) [\alpha(X)\eta(Z) + \beta(Z)\eta(X)] + \delta(\xi)S(Z, X) \\ + \beta(R(X, Z)\xi) + \delta(R(X, \xi)Z).$$

We also have

$$(\nabla_X S)(Z, \xi) = \nabla_X S(Z, \xi) - S(\nabla_X Z, \xi) - S(Z, \nabla_X \xi),$$

which by virtue of (3.8) and (3.6) yields,

$$(\nabla_X S)(Z, \xi) = (n-1) [\nabla_X \eta(Z) - \eta(\nabla_X Z)] + S(Z, \varphi X).$$

The above relation can also be written by means of (3.6) in the form

$$(4.8) \quad (\nabla_X S)(Z, \xi) = -(n-1)g(Z, \varphi X) + S(Z, \varphi X).$$

Hence from (4.7) and (4.8) we get

$$(4.9) \quad (n-1) [\alpha(X)\eta(Z) + \beta(Z)\eta(X)] + \delta(\xi)S(Z, X) + \beta(R(X, Z)\xi) \\ + \delta(R(X, \xi)Z) = -(n-1)g(Z, \varphi X) + S(Z, \varphi X).$$

Putting $X = \xi$ in (4.9) and then using (3.2), (3.8) and (3.9), we obtain (since $R(\xi, \xi)Z = 0$),

$$(4.10) \quad (n-1) [\alpha(\xi) + \delta(\xi)] \eta(Z) + (n-2)\beta(Z) + \beta(\xi)\eta(Z) = 0.$$

Replacing Z by X in (4.10) we have

$$(4.11) \quad (n-1) [\alpha(\xi) + \delta(\xi)] \eta(X) + (n-2)\beta(X) + \beta(\xi)\eta(X) = 0.$$

Again, substituting Z by ξ in (4.9), by virtue of (3.2), (3.8) and (3.9) we get,

$$(4.12) \quad (n-1) [\alpha(X) + \beta(X)] + \delta(X) + (n-2) [\beta(\xi) + \delta(\xi)] \eta(X) = 0.$$

Adding (4.11) and (4.12), we obtain

$$(4.13) \quad (n-1) [\alpha(X) + \beta(X)] + \delta(X) + (n-1) [\alpha(\xi) + \beta(\xi) + \delta(\xi)] \eta(X) \\ + (n-2)\delta(\xi)\eta(X) = 0.$$

Using (4.6) in (4.13), we have

$$(4.14) \quad (n-1) [\alpha(X) + \beta(X)] + \delta(X) + (n-2)\delta(\xi)\eta(X) = 0.$$

Now, putting $Z = \xi$ in (4.5) and using (3.2), (3.8) and (3.9), we get

$$(4.15) \quad (n-1) [\alpha(\xi) + \beta(\xi)] \eta(U) + (n-2)\delta(U) + \delta(\xi)\eta(U) = 0.$$

Replacing U by X in (4.15) we have

$$(4.16) \quad (n-1) [\alpha(\xi) + \beta(\xi)] \eta(X) + (n-2)\delta(X) + \delta(\xi)\eta(X) = 0.$$

Addition of (4.14) and (4.16) gives by virtue of (4.6)

$$(4.17) \quad \alpha(X) + \beta(X) + \delta(X) = 0 \quad \text{for all } X.$$

Hence from (4.17) we can state the following

Theorem 1. *There exist no weakly symmetric K -contact manifolds $M^n(\varphi, \eta, \xi, g)$ ($n > 3$) if $\alpha + \beta + \delta$ is not everywhere zero.*

Since every Sasakian manifold is K -contact we also can state the

Corollary 4.1. *There exist no weakly symmetric Sasakian manifolds $M^n(\varphi, \eta, \xi, g)$ ($n < 2$) if $\alpha + \beta + \delta$ is not everywhere zero.*

The above Corollary 4.1 has been proved by TAMÁSSY and BINH [10].

5. WEAKLY RICCI SYMMETRIC K -CONTACT MANIFOLDS

Let us consider a weakly Ricci-symmetric K -contact manifold (M^n, g) ($n > 3$). By virtue of (3.8), the relation (2.2) gives us

$$(5.1) \quad (\nabla_\xi S)(Y, Z) = \rho(\xi)S(Y, Z) + (n-1)[\mu(Y)\eta(Z) + \nu(Z)\eta(Y)]$$

By virtue of (4.4) and (5.1), we have

$$(5.2) \quad \rho(\xi)S(Y, Z) + (n-1)[\mu(Y)\eta(Z) + \nu(Z)\eta(Y)] = 0.$$

Putting $Y = Z = \xi$ in (5.2) and then using (3.2) and (3.8), we obtain (since $n > 3$)

$$(5.3) \quad \rho(\xi) + \mu(\xi) + \nu(\xi) = 0.$$

Again, replacing Y by ξ in (2.2), by virtue of (4.8) and (3.8) we get,

$$(5.4) \quad \begin{aligned} (n-1)[\rho(X)\eta(Z) + \nu(Z)\eta(X)] + \mu(\xi)S(X, Z) \\ = -(n-1)g(Z, \varphi X) + S(Z, \varphi X). \end{aligned}$$

Putting $Z = \xi$ in (5.4) and then using (3.2) and (3.8), we get

$$(5.5) \quad (n-1)[\rho(X) + \mu(\xi)\eta(X) + \nu(\xi)\eta(X)] = 0.$$

Also, substituting X by ξ in (5.4), by virtue of (3.2) and (3.8) we obtain,

$$(n-1)[\rho(\xi)\eta(Z) + \nu(Z) + \mu(\xi)\eta(Z)] = 0 \quad \text{for all } Z.$$

Replacing Z by X , the above relation reduces to

$$(5.6) \quad (n-1)[\rho(\xi)\eta(X) + \nu(X) + \mu(\xi)\eta(X)] = 0.$$

Adding (5.5) and (5.6), we have by virtue of (5.3),

$$(5.7) \quad \rho(X) + \nu(X) + \mu(\xi)\eta(X) = 0.$$

Now, putting $Z = \xi$ in (5.2) and then using (3.2) and (3.8), we get (since $n > 3$),

$$\mu(Y) + [\rho(\xi) + \nu(\xi)]\eta(Y) = 0, \quad \text{for all } Y,$$

from which follows that

$$(5.8) \quad \mu(X) + [\rho(\xi) + \nu(\xi)]\eta(X) = 0.$$

Adding (5.7) and (5.8) and then using (5.3), we obtain

$$(5.9) \quad \rho(X) + \mu(X) + \nu(X) = 0 \quad \text{for all } X.$$

Hence we can state the following

Theorem 2. *There exists no weakly Ricci symmetric K -contact manifold $M^n(\varphi, \eta, \xi, g)$ ($n > 3$) if $\rho + \mu + \nu$ is not everywhere zero.*

Corollary 5.1. *There exists no weakly Ricci-symmetric Sasakian manifold $M^n(\varphi, \eta, \xi, g)$ ($n > 2$) if $\rho + \mu + \nu$ is not everywhere zero.*

This Corollary has been proved in [10].

6. WEAKLY SYMMETRIC ALMOST EINSTEIN MANIFOLDS

Definition 6.1. A Riemannian manifold (M^n, g) is said to be *almost Einstein* if the Ricci tensor S is of the form

$$(6.1) \quad S(X, Y) = ag(X, Y) + b\omega(X)\omega(Y), \quad \text{for all } X, Y,$$

where a and b are constants, and ω is a non-zero 1-form defined by $\omega(X) = g(X, \tilde{\rho})$.

Now consider a weakly symmetric manifold (M^n, g) which is almost Einstein. Then we have (2.4). Now (4.2) can be written as

$$(6.2) \quad (\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \delta(Z)S(X, Y) \\ = \beta(R(X, Y)Z) + \delta(R(X, Z)Y).$$

Let $g(X, L) = \beta(X)$ and $g(X, M) = \delta(X)$. From (6.1) we get

$$(6.3) \quad (\nabla_X S)(Y, Z) = b[(\nabla_X \omega)(Y)\omega(Z) + \omega(Y)(\nabla_X \omega)(Z)].$$

From (6.2) and (6.3) we obtain

$$(6.4) \quad \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \delta(Z)S(X, Y) + \beta(R(X, Y)Z) \\ + \delta(R(X, Z)Y) = b[(\nabla_X \omega)(Y)\omega(Z) + \omega(Y)(\nabla_X \omega)(Z)].$$

Putting $Y = Z = e_i$ in (6.4) and then taking the sum for $1 \leq i \leq n$, we get

$$\alpha(X)r + 2S(X, L) + 2S(X, M) = 2b \sum_{i=1}^n (\nabla_X \omega)(e_i)\omega(e_i).$$

Using (6.1) in the above equation, we obtain

$$(6.5) \quad \alpha(X)r + 2[ag(X, L) + b\omega(X)\omega(L)] + 2[ag(X, M) + b\omega(X)\omega(M)] \\ = 2b \sum_{i=1}^n (\nabla_X \omega)(e_i)\omega(e_i).$$

The right hand side of (6.5) can be written as

$$2b \sum_{i=1}^n [X\omega(e_i) - \omega(\nabla_X e_i)]\omega(e_i).$$

Let $\{e_i\}$ be an orthonormal basis at $T_\rho M$, $\rho \in M$. Let us translate these e_i parallel from ρ in any direction X_ρ . Then $(\nabla_X e_i)_\rho = 0$. Then the right hand side of (6.5) reduces to

$$2b \sum_{i=1}^n (X\omega(e_i))\omega(e_i) = 2b \sum_{i=1}^n (Xg(e_i, \tilde{\rho}))g(e_i, \tilde{\rho}) \\ = 2b \sum_{i=1}^n g(\nabla_X \tilde{\rho}, e_i)g(e_i, \tilde{\rho}) \\ = 2bg(\nabla_X \tilde{\rho}, \tilde{\rho}) = bXg(\tilde{\rho}, \tilde{\rho}) = bX\|\omega\|^2.$$

Hence (6.5) takes the form

$$r\alpha + 2a\beta + 2a\delta + 2b\omega(L)\tilde{\rho} + 2b\omega(M)\tilde{\rho} = b\|\omega\|^2.$$

Thus we have the following

Theorem 3. *There exists no weakly symmetric almost Einstein manifold if $r\alpha + 2a(\beta + \delta) + 2b(\omega(L)\tilde{\rho} + \omega(M)\tilde{\rho}) - b\|\omega\|^2$ is not everywhere zero.*

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