

ON THE GENERALIZED CESÀRO SUMMABILITY FACTORS

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ABSTRACT. In this paper a general theorem concerning the $\psi - |C, \alpha; \delta|_k$ summability factors of infinite series has been proved.

1. Introduction. A sequence (w_n) of positive numbers is said to be δ -quasi monotone, if $w_n \rightarrow 0$, $w_n > 0$ ultimately and $\Delta w_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see[1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n) . We define A_n^α by identity

$$(1) \quad \sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1}.$$

The sequence-to-sequence transformations given by

$$(2) \quad u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} s_v$$

$$(3) \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v,$$

define the (C, α) means of the sequences (s_n) and (na_n) , respectively.

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$ and $\alpha > -1$, if (see [3])

$$(4) \quad \sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability is the same as $|C, 1|_k$ summability. Let (ψ_n) be a sequence of positive real numbers. We say that the series $\sum a_n$ is said to be summable $\psi - |C, \alpha; \delta|_k$, $k \geq 1$, $\alpha > -1$ and $\delta \geq 0$, if

$$(5) \quad \sum_{n=1}^{\infty} \psi_n^{\delta k + k - 1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty.$$

But since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [4]) condition (5) can also be written as

$$(6) \quad \sum_{n=1}^{\infty} \psi_n^{\delta k + k - 1} n^{-k} |t_n^\alpha|^k < \infty.$$

If we take $\delta = 0$ and $\psi_n = n$ (resp. $\delta = 0$, $\alpha = 1$ and $\psi_n = n$), then $\psi - |C, \alpha; \delta|_k$ summability is the same as $|C, \alpha|_k$ (resp. $|C, 1|_k$) summability.

Remark. Since (ψ_n) is a sequence of positive real numbers the summability

method $\psi - |C, \alpha; \delta|_k$ is a new method and general than the $|C, \alpha; \delta|_k$ summability method. On the other hand $|C, \alpha; \delta|_k$ and $\psi - |C, \alpha; \delta|_k$ summability methods are different from each other. That is they have got different summability fields. Therefore, we take the sequence (ψ_n) instead of n .

2. The following theorem is known.

Theorem A ([2]). Let t_n^α be the n -th Cesàro mean of order α , with $\alpha \geq 1$, of the sequence (na_n) such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$ and let $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ -quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$, $\sum B_n \log n$ is convergent and $|\Delta \lambda_n| \leq |B_n|$ for all n .

$$(7) \quad \sum_{n=1}^m |\Delta(n^\alpha)| |B_{n+1}| \log n = O(1),$$

$$(8) \quad \sum_{n=1}^m \frac{1}{n} |t_n^\alpha|^k = O(\log m) \text{ as } m \rightarrow \infty,$$

then the series $\sum a_n \lambda_n$ is summable $|C, \alpha|_k$, $k \geq 1$.

3. The aim of this paper is to generalize Theorem A in the following form.

Theorem. Let $k \geq 1$ and $\delta \geq 0$. Let t_n^α be the n -th Cesàro mean of order α , with $\alpha \geq 1$, of the sequence (na_n) such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$ and let $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ -quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$, $\sum B_n \log n$ is convergent, $|\Delta \lambda_n| \leq |B_n|$ for all n and that condition (7) of Theorem A is satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \psi_n^{\delta k + k - 1})$ is non-increasing and

$$(9) \quad \sum_{n=1}^m \psi_n^{\delta k + k - 1} n^{-k} |t_n^\alpha|^k = O(\log m) \text{ as } m \rightarrow \infty,$$

then the series $\sum a_n \lambda_n$ is summable $\psi - |C, \alpha; \delta|_k$.

If we take $\delta = 0$, $\epsilon = 1$ and $\psi_n = n$ in this theorem, then we get Theorem A.

4. We need the following lemmas for the proof of our theorem.

Lemma 1 ([5]). If $\sigma > \delta > 0$, then

$$(10) \quad \sum_{n=v+1}^m \frac{A_{n-v}^{\delta-1}}{A_n^\sigma} = \sum_{n=v+1}^m \frac{(n-v)^{\delta-1}}{n^\sigma} = O(v^{\delta-\sigma}) \text{ as } m \rightarrow \infty.$$

Lemma 2 ([2]). Let $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Suppose that there exists a sequence of numbers (B_n) which is δ -quasi monotone with $\sum B_n \log n$ is convergent and $|\Delta \lambda_n| \leq |B_n|$ for all n , then

$$(11) \quad |\lambda_n| \log n = O(1) \text{ as } n \rightarrow \infty.$$

Lemma 3 ([2]). Let $\alpha \geq 1$. If (B_n) is δ -quasi monotone with $\sum n^\alpha \delta_n \log n < \infty$ and $\sum B_n \log n$ is convergent, then

$$(12) \quad m^\alpha B_m \log m = O(1) \text{ as } m \rightarrow \infty,$$

$$(13) \quad \sum_{n=1}^{\infty} n^\alpha |\Delta B_n| \log n < \infty.$$

Lemma 4 ([2]). Let t_n^α be the n -th Cesàro mean of order α , with $\alpha \geq 1$, of the sequence (na_n) such that $a_n \geq 0$ for all $n \geq 1$ whenever $\alpha > 1$. If $n \geq v$, then

$$(14) \quad \left| \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p \right| \leq A_{n-v}^{\alpha-1} A_v^\alpha |t_v^\alpha|.$$

5. Proof of the Theorem. Let (T_n^α) be the n -th (C, α) , with $\alpha \geq 1$, means of the sequence $(na_n\lambda_n)$. Then, by (3), we have

$$(15) \quad T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \lambda_v.$$

Using Abel's transformation, we get

$$\begin{aligned} T_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \\ &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p + \lambda_n t_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \text{ say.} \end{aligned}$$

Since

$$|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \leq 2^k (|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k),$$

to complete the proof of the theorem, it is sufficient to show that

$$(16) \quad \sum_{n=1}^{\infty} \psi_n^{\delta k+k-1} n^{-k} |T_{n,r}^\alpha|^k < \infty \text{ for } r = 1, 2, \text{ by (6).}$$

Firstly, when $k > 1$, using Lemma 4 and after applying Hölder's inequality with indices k and k' , where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} |T_{n,1}^\alpha|^k &= \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} \left| \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p \right|^k \\ &\leq \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} (A_n^\alpha)^{-k} \left\{ \sum_{v=1}^{n-1} |B_v| A_v^\alpha A_{n-v}^{\alpha-1} |t_v^\alpha| \right\}^k \\ &= O(1) \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} (A_n^\alpha)^{-k} \left\{ \sum_{v=1}^{n-1} v^\alpha |B_v| A_{n-v}^{\alpha-1} |t_v^\alpha| \right\}^k \\ &= O(1) \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} A_n^\alpha \sum_{v=1}^{n-1} (v^\alpha |B_v|)^k A_{n-v}^{\alpha-1} |t_v^\alpha|^k \\ &\quad \times \left\{ \sum_{v=1}^{n-1} \frac{A_{n-v}^{\alpha-1}}{A_n^\alpha} \right\}^{k-1} \\ &= O(1) \sum_{v=1}^m (v^\alpha |B_v|)^{k-1} (v^\alpha |B_v|) |t_v^\alpha|^k \sum_{n=v}^{m+1} \frac{\psi_n^{\delta k+k-1} A_{n-v}^{\alpha-1}}{n^k A_n^\alpha} \\ &= O(1) \sum_{v=1}^m v^\alpha |B_v| |t_v^\alpha|^k \sum_{n=v}^{m+1} \frac{\psi_n^{\delta k+k-1} n^{\epsilon-k} (n-v)^{\alpha-1}}{n^{\alpha+\epsilon}} \\ &= O(1) \sum_{v=1}^m v^\alpha |B_v| |t_v^\alpha|^k \psi_v^{\delta k+k-1} v^{\epsilon-k} \sum_{n=v}^{m+1} \frac{(n-v)^{\alpha-1}}{n^{\alpha+\epsilon}} \\ &= O(1) \sum_{v=1}^m v^\alpha |B_v| |t_v^\alpha|^k \psi_v^{\delta k+k-1} v^{-k}, \end{aligned}$$

by Lemma 1. Thus

$$\sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} |T_{n,1}^\alpha|^k = O(1) \sum_{v=1}^{m-1} \Delta(v^\alpha |B_v|) \sum_{p=1}^v \psi_p^{\delta k+k-1} p^{-k} |t_p^\alpha|^k$$

$$\begin{aligned}
& + O(1)m^\alpha |B_m| \sum_{v=1}^m \psi_v^{\delta k+k-1} v^{-k} |t_v^\alpha|^k \\
& = O(1) \sum_{v=1}^{m-1} \Delta(v^\alpha |B_v|) \log v + O(1)m^\alpha |B_m| \log m \\
& = O(1) \sum_{v=1}^{m-1} v^\alpha |\Delta B_v| \log v + O(1) \sum_{v=1}^{m-1} |\Delta(v^\alpha)| |B_{v+1}| \log v \\
& + O(1)m^\alpha |B_m| \log m = O(1) \text{ as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma 3.

Again, since $|\lambda_n| = O(1)$, we have that

$$\begin{aligned}
\sum_{n=1}^m \psi_n^{\delta k+k-1} n^{-k} |T_{n,2}^\alpha|^k & = \sum_{n=1}^m \psi_n^{\delta k+k-1} n^{-k} |\lambda_n t_n^\alpha|^k \\
& = \sum_{n=1}^m \psi_n^{\delta k+k-1} n^{-k} |\lambda_n|^{k-1} |\lambda_n| |t_n^\alpha|^k \\
& = O(1) \sum_{n=1}^m \psi_n^{\delta k+k-1} n^{-k} |\lambda_n| |t_n^\alpha|^k \\
& = O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{p=1}^n \psi_p^{\delta k+k-1} p^{-k} |t_p^\alpha|^k \\
& + O(1) |\lambda_m| \sum_{n=1}^m \psi_n^{\delta k+k-1} n^{-k} |t_n^\alpha|^k \\
& = O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| \log n + O(1) |\lambda_m| \log m \\
& = O(1) \sum_{n=1}^{m-1} |B_n| \log n + O(1) |\lambda_m| \log m \\
& = O(1) \text{ as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma 2.

Therefore, we get (16). This completes the proof of the Theorem.

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