

ON THE LOCAL LIMIT THEOREM FOR GENERAL LATTICE DISTRIBUTION

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ABSTRACT. The scheme of series of lattice casual vectors analogical to the probable is considered in the paper. A local limit theorem have been received, and the indicated directions of its use in differential methods theory of mathematical physics problems solving were received.

A local limit theorem for lattice distributions, and their quasiprobable analogues are successfully used in the approximate solution of some kinds of mathematical physics problems [1-3].

Quasiprobable distribution on the plain (according to Y. P. Studnyev [4]) on the whole is the complex number set $\{p(k, l)\}$, for which

$$\sum_{k,l} p(k, l) = 1; \quad \sum_{k,l} |p(k, l)| < +\infty.$$

Let $\{(\xi_{nn}, \eta_{nn})\}$ be sequence of a mutually independent series of probable vectors (first index - series number, the second is changed from 1 to n), which are equally distributed on $\{kh_1, lh_2\}$ lattice, and let $\{p(n, k, l)\}$ be a quasiprobable vector distribution in series frames in the meaning of correspondence $p(n, k, l) = P\{(\xi_{nj}, \eta_{nj}) = (kh_1, lh_2)\}$, where $k, l \in Z; h_1, h_2 > 0; j = \overline{1, n}$. Then

$$w(t, s) = \sum_{k,l} e^{i(tk h_1 + sl h_2)} p(n, k, l)$$

is a Fourier-Stieltjes distribution transformation $\{p(n, k, l)\}$.

As in the theory of probabilities, the formula of reverse is being proved:

$$p(n, k, l) = \frac{h_1 h_2}{4\pi^2} \iint_D e^{-i(tk h_1 + sl h_2)} w(t, s) dt ds,$$

where D rectangle is in the form of $\left[-\frac{\pi}{h_1}, \frac{\pi}{h_1}, -\frac{\pi}{h_2}, \frac{\pi}{h_2}\right]$.

If $\{p_n(k, l)\}$ is $\left(\sum_{j=1}^n \xi_{nj}, \sum_{j=1}^n \eta_{nj}\right)$ sum vector distribution, then the reverse formula is in

$$(1) \quad P_n(n, k, l) = \frac{h_1 h_2}{4\pi^2} \iint_D e^{-i(tk h_1 + sl h_2)} w(t, s) dt ds,$$

where $\{P_n(k, l)\}$ distribution is n -divisible fold of the set $\{p_n(k, l)\}$ with itself. Let, further

$$\alpha_{r,q} = \sum_{k,l} (kh_1)^r (lh_2)^q p(n, k, l)$$

be an initial moment with (r, q) order of $\{p(n, k, l)\}$ distribution.

Let us consider two auxiliary lemmas.

Lemma 1. *Let for the casual vector with distribution conditions exist :*

- 1) $\alpha_{1,0} = \alpha_{0,1} = \alpha_{3,0} = \alpha_{2,1} = \alpha_{1,2} + \alpha_{0,3} = \alpha_{3,1} = \alpha_{2,2} = \alpha_{1,3} = 0$;
 - 2) $\alpha_{2,0} = -\frac{2a}{\sqrt{n}}, \alpha_{1,1} = -\frac{2b}{\sqrt{n}}, \alpha_{0,2} = -\frac{2c}{\sqrt{n}}, at^2 + bts + cs^2$ is a positively determined square form;
 - 3) $\alpha_{4,0} = \alpha_{0,4} = 4!A, A > 0$;
 - 4) $a_{r,q}$ exist, when $r = q = 5$.
- Then

$$(2) \quad \left| w^n \left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}} \right) - e^{at^2 + bts + cs^2 - A(t^4 + s^4)} dt ds \right| = O \left(\frac{|t|^5 + |s|^5}{\sqrt[4]{n}} \right), n \rightarrow \infty$$

correlation comes true.

Proof. We should note, that the role of moments in apportion is the same, as for characteristic functions in the analogical situation. That is why

$$w(t, s) = 1 + \frac{1}{\sqrt{n}} (at^2 + bts + cs^2) - A(t^4 + s^4) + O(|t|^5 + |s|^5), t, s \rightarrow 0.$$

But then

$$w \left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}} \right) = 1 + \frac{at^2 + bts + cs^2}{n} - \frac{A(t^4 + s^4)}{n} + O \left(\frac{|t|^5 + |s|^5}{n^4 \sqrt[4]{n}} \right), \quad n \rightarrow \infty;$$

or

$$w \left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}} \right) = e^{\frac{1}{n}(at^2 + bts + cs^2) - \frac{A(t^4 + s^4)}{n}} + O \left(\frac{|t|^5 + |s|^5}{\sqrt[4]{n}} \right), \quad n \rightarrow \infty$$

for any fix couple $(t, s) \in R_2$. Hence, it follows, after the n -th power production of both parts of the last equality we receive (2). Then Lemma 1 is proved.

Let's present $w(t, s)$ into

$$w(t, s) = w_0(t, s) + \frac{1}{\sqrt{n}} (at^2 + bts + cs^2)$$

form. □

Lemma 2. *Let in $\{\xi_{nn}, \eta_{nn}\}$ series sequence, which are mutually independent and equally distributed within the limits of lattice casual vector series for every vector in series Lemma 1 conditions and condition : $|w_0(t, s)| < 1$, when $(t, s) \in D \setminus \{0, 0\}$, are executed. Then $A_0 > 0$ exists that in $\sqrt[4]{n}D = \left[-\frac{\sqrt[4]{n\pi}}{h_1}, \frac{\sqrt[4]{n\pi}}{h_1}, -\frac{\sqrt[4]{n\pi}}{h_2}, \frac{\sqrt[4]{n\pi}}{h_2} \right]$ rectangle execute*

$$(3) \quad \left| w^n \left(\frac{t}{\sqrt[4]{n}}, \frac{s}{\sqrt[4]{n}} \right) \right| \leq e^{at^2 + bts + cs^2 - A_0(t^4 + s^4)}.$$

Proof. In the work [3] it has been shown that by Lemma 2 conditions $A_0 > 0$ would exist that $|w_0(t, s)| \leq e^{-A_0(t^4 + s^4)}, (t, s) \in D$.

Then in D executes

$$|w(t, s)| \leq |w_0(t, s)| + \frac{at^2 + bts + cs^2}{\sqrt{n}} \leq e^{\frac{at^2 + bts + cs^2}{\sqrt{n}} - A_0(t^4 + s^4)},$$

whence directly (3) comes out. $\square$

These lemmas permit to prove such a theorem.

Theorem. *Let in $\{(\xi_{nn}, \eta_{nn})\}$ series sequences, which are mutually independent and equally distributed within the limits of lattice casual vector series for every vector in series conditions :*

- 1) $\alpha_{1,0} = \alpha_{0,1} = \alpha_{3,0} = \alpha_{2,1} = \alpha_{1,2} = \alpha_{0,3} = \alpha_{3,1} = \alpha_{2,2} = \alpha_{1,3} = 0$;
 - 2) $\alpha_{2,0} = -\frac{2a}{\sqrt{n}}, \alpha_{1,1} = -\frac{2b}{\sqrt{n}}, at^2 + bts + cs^2$ is a positively determined square form;
 - 3) $\alpha_{4,0} = \alpha_{0,4} = 4!A, A > 0$;
 - 4) $\alpha_{r,q}$ exist, when $r + q = 5$;
 - 5) $|w_0(t, s)| < 1$, when $(t, s) \in D \setminus \{0, 0\}$.
- Then evenly on $k \in Z$ by $n \rightarrow \infty$:

$$(4) \quad \sqrt{n} \left(\frac{p_n(k, l)}{h_1 h_2} - \frac{1}{4\pi^2} \iint e^{-i(tk h_1 + sl h_2) + n \left(\frac{at^2 + bts + cs^2}{\sqrt{n}} - A(t^4 + s^4) \right)} dt ds \right) \rightarrow 0.$$

Proof. Use the reverse formula (1) and in integrals of the left part of (4) correlation defined through R_n , execute the substitute: $t \rightarrow \frac{t}{4\sqrt{n}}, s \rightarrow \frac{s}{4\sqrt{n}}$.

We receive

$$4\pi^2 R_n = \iint_{4\sqrt{n}D} e^{-i \left(\frac{tk h_1}{4\sqrt{n}} + \frac{sl h_2}{4\sqrt{n}} \right)} w^n \left(\frac{t}{4\sqrt{n}}, \frac{s}{4\sqrt{n}} \right) dt ds - \\ - \iint e^{-i \left(\frac{tk h_1}{4\sqrt{n}} + \frac{sl h_2}{4\sqrt{n}} \right) + at^2 + bts + cs^2 - A(t^4 + s^4)} dt ds = I_1 + I_2 - I_3,$$

where

$$I_1 = \iint_{\Delta} e^{-i \left(\frac{tk h_1}{4\sqrt{n}} + \frac{sl h_2}{4\sqrt{n}} \right)} \left(w^n \left(\frac{t}{4\sqrt{n}}, \frac{s}{4\sqrt{n}} \right) - e^{at^2 + bts + cs^2 - A(t^4 + s^4)} \right) dt ds,$$

$$I_2 = \iint_{4\sqrt{n}D \setminus \Delta} e^{-i \left(\frac{tk h_1}{4\sqrt{n}} + \frac{sl h_2}{4\sqrt{n}} \right)} w^n \left(\frac{t}{4\sqrt{n}}, \frac{s}{4\sqrt{n}} \right) dt ds,$$

$$I_3 = \iint_{R_2 \setminus \Delta} e^{-i \left(\frac{tk h_1}{4\sqrt{n}} + \frac{sl h_2}{4\sqrt{n}} \right) + at^2 + bts + cs^2 - A(t^4 + s^4)} dt ds;$$

$$\Delta = \{(t, s) : |t| \leq n^\lambda, 0 < \lambda < \frac{1}{28}\}.$$

Integral I_1 estimation comes out of Lemma 1. Really, according to (2) $B > 0$ exists, that

$$\left| w^k \left(\frac{t}{4\sqrt{n}}, \frac{s}{4\sqrt{n}} \right) - e^{at^2 + bts + cs^2 - A(t^4 + s^4)} \right| \leq B \left(\frac{|t|^5 + |s|^5}{4\sqrt{n}} \right), (t, s) \in \Delta.$$

Then

$$|I_1| \leq \frac{B}{4\sqrt{n}} \iint_{\Delta} (|t|^5 + |s|^5) dt ds = \frac{4B}{3} n^{7\lambda - \frac{1}{4}} = \frac{4B}{3} n^{-7(\frac{1}{28} - \lambda)}.$$

So as $\lambda < \frac{1}{28}$, then $I_1 \rightarrow 0$ by $n \rightarrow \infty$.

Out of the Lemma 2 the I_2 estimation comes out. Using (3), we receive

$$|I_2| \leq \iint_{4\sqrt{n}D \setminus \Delta} e^{at^2 + bts + cs^2 - A(t^4 + s^4)} dt ds.$$

The positively determined square form permits an upper estimation:

$$at^2 + bts + cs^2 \leq a_0 (t^2 + s^2), \quad a_0 > 0.$$

Then

$$|I_2| \leq \iint_{4\sqrt{n}D \setminus \Delta} e^{a_0(t^2+s^2)-A(t^4+s^4)} dt ds \leq 4 \left(\int_{n^\lambda}^{\infty} e^{a_0 t^2 - A_0 t^4} dt \right)^2.$$

We should take into consideration that by sufficiently large n for $t \geq n^\lambda$

$$4A_0 t^3 - 2a_0 t > 4A_0 n^{3\lambda} - 2a_0 n^\lambda,$$

I_2 integral permits the further estimation :

$$|I_2| \leq \frac{1}{(2A_0 n^{3\lambda} - a_0 n^\lambda)^2} \int_{n^\lambda}^{\infty} \left((4A_0 t^3 - 2a_0 t) e^{a_0 t^2 - A_0 t^4} dt \right)^2 = \left(\frac{e^{a_0 n^{2\lambda} - A_0 n^{4\lambda}}}{2A_0 n^{3\lambda} - a_0 n^\lambda} \right)^2,$$

therefore, by $n \rightarrow \infty$ $I_2 \rightarrow 0$.

I_3 integral estimation is analogical to the second stage of I_2 integral estimation. The Theorem is proved. $\square$

The received theorem could easy be generalized in case of quasiprobable lattice distributions with Fourier-Stieltjes transformation in

$$w(t, s) = e^{\Psi_2(t,s) + \Psi_4(t,s) + \dots + \Psi_{2q-2}(t,s) - \Psi_{2q}(t,s)}$$

form, where $\Psi_2(t, s)$, $\Psi_4(t, s)$, $\Psi_{2q}(t, s)$ are positively-determined forms with orders indicated by indexes.

Such generalization could be used to the approximate problems solving linked with the evolutionary equation in the form of:

$$\begin{aligned} \frac{\partial u(x, y, \tau)}{\partial \tau} = & \left((-1)^{q+1} \Psi_{2q} \left(i \frac{\partial}{\partial x}, i \frac{\partial}{\partial y} \right) - (-1)^q \Psi_{2q-2} \left(i \frac{\partial}{\partial x}, i \frac{\partial}{\partial y} \right) - \right. \\ & \left. - \dots - \Psi_2 \left(i \frac{\partial}{\partial x}, i \frac{\partial}{\partial y} \right) \right) u(x, y, \tau) \end{aligned}$$

according to the scheme, for example, in [3].

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