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ON A SIMULTANEOUS APPROXIMATION PROBLEM CONCERNING BINARY RECURRENCES

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Dedicated to Professor Árpád Varecza on his 60th birthday

Abstract. Let $R_n \quad (n = 0, 1, 2, \ldots)$ be a second order linear recursive sequence of rational integers defined by $R_n = AR_{n-1} + BR_{n-2}$ for $n > 1$, where A and B are integers and the initial terms are $R_0 = 0,\ R_1 = 1$. It is known, that if α, β are the roots of the equation $x^2 - Ax - B = 0$ and $|\alpha| > |\beta|$, then $R_{n+1}/R_n \longrightarrow \alpha$ as $n \longrightarrow \infty$. Approximating α with the rational number R_{n+1}/R_n, it was shown that $\left|\alpha - \frac{R_{n+1}}{R_n}\right| < \frac{1}{c \cdot |R_n|^2}$ holds with a constant $c > 0$ for infinitely many n if and only if $|B| = 1$. In this paper we investigate the quality of the approximation of α and α^s by the rational numbers R_{n+1}/R_n and R_{n+s}/R_n simultaneously.

Introduction

Let A and B be fixed non-zero integers and let $\{R_n\}_{n=0}^{\infty}$ be a second order linear recursive sequence of rational integers defined by the recursion

$$R_n = AR_{n-1} + BR_{n-2} \quad (n > 1)$$

with initial terms $R_0 = 0$ and $R_1 = 1$. Denote by α and β the roots of the characteristic equation

$$x^2 - Ax - B = 0$$

of the sequence and suppose that $|\alpha| \geq |\beta|$. We suppose that the sequence $\{R_n\}$ is non degenerate, i.e. α/β is not a root of untity. In this case the terms of the sequence can be expressed as

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1}$$

for any $n \geq 0$.

If $D = A^2 + 4B > 0$, then α and β are real numbers and $|\alpha| > |\beta|$. It implies that

$$\frac{R_{n+1}}{R_n} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \alpha \cdot \frac{1 - (\beta/\alpha)^{n+1}}{1 - (\beta/\alpha)^n} \longrightarrow \alpha \quad \text{as} \quad n \longrightarrow \infty$$

and so α can be approximated by rational numbers R_{n+1}/R_n. For the quality of this approximation in [2] we proved that the inequality

$$\left|\alpha - \frac{R_{n+1}}{R_n}\right| < \frac{1}{c \cdot |R_n|^2}$$

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with some $c > 0$ holds for infinitely many n if and only if $|B| = 1$, furthermore in the case $|B| = 1$ the best approximation constant is $c = \sqrt{D}$. It was also proved that if $|B| = 1$ and $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{D}q^2}$ with a rational number p/q, then $p/q = R_{n+1}/R_n$ for some n.

In this paper we deal with a simultaneous approximation problem. Let γ_1 and γ_2 be irrational numbers. It is known that there are infinitely many triples p_1, p_2, q of rational integers and a constant $c > 0$ such that

$$\left|\gamma_1 - \frac{p_1}{q}\right| < \frac{1}{c \cdot q^{3/2}} \quad \text{and} \quad \left|\gamma_2 - \frac{p_2}{q}\right| < \frac{1}{c \cdot q^{3/2}}$$

hold simultaneously and the order $\frac{3}{2}$ of the approximation is the best possibility in general. In the next section we show that if $\gamma_1 = \alpha$ and $\gamma_2 = \alpha^s$ where s is a positive integer then the order of their simultaneous approximation can be 2.

The real case

In this section we investigate the case when $D = A^2 + 4B > 0$, so α, β are real numbers and $|\alpha| \neq |\beta|$.

Theorem 1. *Let $\{R_n\}$ be a second order linear recursive sequence defined in the Introduction and let $s \geq 2$ be a positive integer. Suppose that $D > 0$, $|\alpha| > |\beta|$ and $|B| = 1$. Then there is a constant $c_0 > 0$ such that the inequalities*

$$\left|\alpha - \frac{R_{n+1}}{R_n}\right| < \frac{1}{c_0 R_n^2} \quad \textit{and} \quad \left|\alpha^s - \frac{R_{n+s}}{R_n}\right| < \frac{1}{c_0 \cdot R_n^2}$$

hold simultaneously for infinitely many positive integer n.

Proof. For an integer $k \geq 1$ by (1) we have

$$\left|\alpha^k - \frac{R_{n+k}}{R_n}\right| = \left|\alpha^k - \frac{\alpha^{n+k} - \beta^{n+k}}{\alpha^n - \beta^n}\right| = \left|\frac{\beta^k\beta^n - \alpha^k\beta^n}{\alpha^n - \beta^n}\right| =$$

$$|\beta|^n \cdot \left|\frac{\alpha^k - \beta^k}{\alpha^n - \beta^n}\right| = |\beta|^n \frac{|R_k|}{|R_n|} = |\beta|^n \left|\frac{\alpha^n - \beta^n}{\alpha - \beta}\right| \cdot \frac{|R_k|}{R_n^2} =$$

$$|\alpha\beta|^n \cdot \frac{1}{|\alpha - \beta|} \cdot \left|1 - \left(\frac{\beta}{\alpha}\right)^n\right| \cdot \frac{|R_k|}{R_n^2}. \tag{2}$$

But $\left|\frac{\beta}{\alpha}\right| < 0$, $|\alpha - \beta| = \sqrt{D}$ and $|\alpha\beta| = 1$ since $|B| = 1$, so $(\beta/\alpha)^n \longrightarrow 0$ as $n \longrightarrow \infty$ and by (2)

$$\left|\alpha^k - \frac{R_{n+k}}{R_n}\right| < \frac{|R_k|}{\sqrt{D}} \cdot \frac{1}{R_n^2}$$

for any $k \geq 1$ and for infinitely many positive integer n (for any n if $\beta/\alpha > 0$ and for any even n if $\beta/\alpha < 0$). From this inequality the theorem follows with

$$c_0 = \min\left(\frac{\sqrt{D}}{|R_1|}, \frac{\sqrt{D}}{|R_s|}\right) = \frac{\sqrt{D}}{|R_s|}.$$

We note that some other approximation results was obtained by F. Mátyás [6] and B. Zay [7] concerning general recurrences. □

Complex case

Now let $D < 0$. In this case α and β are not real complex conjugate numbers with $\left|\frac{\beta}{\alpha}\right| = 1$ and we can conclude in (2) only that $0 < |1 - (\beta/\alpha)^n| < 2$, so $\lim\limits_{n\to\infty} \frac{R_{n+k}}{R_n}$ does not exist for any $k \geq 1$. We can approximate only the powers of $|\alpha|$ instead of the powers of α and the quality of these approximations is much weaker than in Theorem 1. In [3] and [4] with R. F. Tichy we proved that there are positive constants c_1 and c_2, depending only on the sequence $\{R_n\}$, such that

$$\left| |\alpha| - \left|\frac{R_{n+1}}{R_n}\right| \right| < \frac{1}{n^{c_1}} \tag{3}$$

for infinitely many n but

$$\left| |\alpha| - \left|\frac{R_{n+1}}{R_n}\right| \right| > \frac{1}{n^{c_2}}$$

for all suficiently large n. It shows that, apart from the constant c_1, (3) is the best possibility to approximate $|\alpha|$ by rational numbers of the form $|R_{n+1}/R_n|$. Now we prove:

Theorem 2. *For any non-degenerate second order linear recurrence $\{R_n\}$, defined in the Introduction, for which $D < 0$, there are constants $c_4 > c_3 > 0$ such that the inequalities*

$$\left| |\alpha| - \left|\frac{R_{n+1}}{R_n}\right| \right| < \frac{1}{n^{c_3}} \tag{4}$$

and

$$\left| |\alpha|^s - \left|\frac{R_{n+s}}{R_n}\right| \right| < \frac{1}{n^{c_3}} \tag{5}$$

hold simultaneously for infinitely many pairs $s > 1,\ n$ of positive integers, but

$$\left| |\alpha|^s - \left|\frac{R_{n+s}}{R_n}\right| \right| > \frac{1}{n^{c_4}} \tag{6}$$

for any given s and sufficiently large n.

For the proof of Theorem 2 we need some auxiliary results.

First we recall some results concerning distribution properties of sequences of real numbers modulo 1. Let (x_n) $(n = 1, 2, \ldots)$ be a sequence of real numbers. Denote by $\{x_n\}$ the fractional part of a term x_n and denote I intervals for which $I \in [0, 1]$. For a positive integer N let $A_N(x_n, I)$ be the number of indices n with $1 \leq n \leq N$ such that the fractional part of x_n is contained in the interval I, i.e.

$$A_N(x_n, I) = \operatorname{card}\{n \leq N : \{x_n\} \in I\}.$$

then the discrepancy of the sequence x_n is defined by

$$D_N(x_n) = \sup_I \left| \frac{A_N(x_n, I)}{N} - |I| \right|,$$

where the supremum is taken over all subintervals I of $[0, 1]$.

From the definition of $D_N(x_n)$ it follows that if $|I| \geq 2D_N(x_n)$, then there exists an integer n with $1 \leq n \leq N$ such that $\{x_n\} \in I$. For a special sequence the following estimation hold.

Lemma 1. *Let $\gamma = e^{2\pi\theta i}$ be a complex number, where $|\gamma| = 1$ and $0 < \theta < 1$ is an irrational number. Then the discrepancy of the sequence $(x_n) = (n\theta)$ satisfies the estimation*

$$D_N(x_n) \leq N^{-\delta}$$

for any sufficiently large N, where δ (> 0) depends only on γ.

Proof. The lemma follows from a more general theorem of [4], but it can be proved directly using Theorem 2.5 of [5], p. 112. We need another result, too. □

Lemma 2. *Let*
$$\Lambda = b_1 \cdot \log \omega_1 + \cdots + b_t \cdot \log \omega_t,$$
where b_i's are rational integers and ω_i's are algebraic numbers different from 0 and 1. Suppose that not all of the b_i's are 0 and that the logarithms mean their principal walues. Assume that $\max(|b_i|) \le B \quad (B \ge 4)$, ω_i *has height at most* M_i (≥ 4) *and that the field generated by the ω_i's over the rational numbers has degree at most d. If $\Lambda \ne 0$, then*
$$|\Lambda| > B^{-C\Omega\cdot\log\Omega'},$$
where $\Omega = \log M_1 \cdot \log M_2 \cdots \log M_t$, $\quad \Omega' = \Omega/\log M_t$ *and C is an effectively computable positive constant depending only on t and d.*

Proof. It is a result of A. Baker, see in [1]. □

Proof of Theorem 2. α and β are conjugate complex numbers so we can write
$$\beta = r\cdot e^{\pi\theta i}, \quad \alpha = r\cdot e^{-\pi\theta i} \quad \text{and} \quad \frac{\beta}{\alpha} = e^{2\pi\theta i},$$
where $0 < \theta < 1$ and θ is an irrational number since β/α is not a root of unity. By (1), for any $k \ge 1$ we have

$$\left|\frac{R_{n+k}}{R_n}\right| = \frac{|\alpha^{n+k}(1-(\beta/\alpha)^{n+k})|}{|\alpha^n(1-(\beta/\alpha)^n)|} = |\alpha|^k \left|\frac{1-e^{2\pi(n+k)\theta i}}{1-e^{2\pi n\theta i}}\right|. \tag{7}$$

Let N be a positive integer large enought and denote by D_N the discrepancy of the sequence $(x_n) = (n\theta)$. Then, as we have seen above, there are integers m_{k1} and m_{k2} with $1 \le m_{k1} < m_{k2} \le N$ such that
$$|m_{k1}\theta - p - \left(1-\frac{\theta}{2}\right)| < 2D_N$$
and
$$|m_{k2}\theta - q - \left(1+\frac{\theta}{2}\right)| < 2D_N,$$
where p and q are suitable integers. From these inequalities, using the notation $z = e^{2\pi\left(1-\frac{\theta}{2}\right)i}$,

$$e^{2\pi m_{k1}\theta i} = e^{2\pi\left(1-\frac{\theta}{2}+\varepsilon_0\right)i} = z\cdot e^{2\pi\varepsilon_0 i},$$

$$e^{2\pi(m_{k1}+1)\theta i} = e^{2\pi\left(1+\frac{\theta}{2}+\varepsilon_0\right)i} = \overline{z}\cdot e^{2\pi\varepsilon_0 i}$$

and

$$e^{2\pi m_{k2}\theta i} = e^{2\pi\left(1+\frac{\theta}{2}+\varepsilon_1\right)i} = \overline{z}\cdot e^{2\pi\varepsilon_1 i}$$

follows, where $|\varepsilon_0|, |\varepsilon_1| = O(D_N)$. So by (7), with $m_{k1} = n$ and $m_{k2} = n+s$, we obtain the estimations

$$\left||\alpha| - \left|\frac{R_{n+1}}{R_n}\right|\right| = |\alpha|\cdot\left|1-\left|\frac{1-\overline{z}\cdot e^{2\pi\varepsilon_0 i}}{1-z\cdot e^{2\pi\varepsilon_0 i}}\right|\right| = |\alpha|\cdot O(D_N) \tag{8}$$

and

$$\left||\alpha^s| - \left|\frac{R_{n+s}}{R_n}\right|\right| = |\alpha|^s\cdot\left|1-\left|\frac{1-\overline{z}\cdot e^{2\pi\varepsilon_1 i}}{1-z\cdot e^{2\pi\varepsilon_0 i}}\right|\right| = |\alpha|^s\cdot O(D_N). \tag{9}$$

From (8) and (9) the inequalities (4) and (5) follow, since $O(D_N) < \frac{1}{n^{c_3'}}$ by Lemma 1. for any $c_3' < \delta$. If we choose another N' $(> N)$ which is sufficiently large, then we obtain another pair of integers m'_{k1}, m'_{k2} and so the existence of infinitely many integers n, s can be concluded.

Now we prove inequality (6). Similary as above we obtain that

$$\left| |\alpha|^s - \left| \frac{R_{n+s}}{R_n} \right| \right| = |\alpha|^s \cdot \left| 1 - \left| \frac{1 - (\beta/\alpha)^{n+s}}{1 - (\beta/\theta)^n} \right| \right| . \tag{10}$$

Let z be a complex number defined by

$$z = e^{(\pi - \pi s\theta)i}.$$

Then for any integers $s \geq 1$ and n we have

$$\left(\frac{\beta}{\alpha} \right)^n = e^{2\pi n\theta i} = z \cdot e^{\lambda i},$$

where $0 < \lambda < 2\pi$ and

$$\lambda = 2\pi n\theta - \pi + \pi s\theta - 2\pi k =$$
$$(2n + s)\pi\theta - (2k + 1)\pi =$$

$$(2n + s) \cdot \arg(\beta) - (2k + 1) \cdot \arg(-1) =$$
$$(2n + s) \cdot \log \beta - (2n + s) \cdot \log |\beta| - (2k + 1) \cdot \log(-1)$$

with some integer $k < n+s$. β, $|\beta|$ and -1 are algebraic numbers of degree at most 4, furthermore $\lambda \neq 0$ since θ is an irrational number, so by Lemma 2 we obtain the inequality

$$|\lambda| > n^{-c_4}$$

where $c_4 > 0$ depends on s and the sequences $\{R_n\}$. It can be similary proved that

$$|\pi - \lambda| > n^{-c_5}.$$

These inequalities imply that

$$|Im(e^{\lambda i}| > n^{-c_6}. \tag{11}$$

By (10) we get

$$\left| |\alpha|^s - \left| \frac{R_{n+s}}{R_n} \right| \right| = |\alpha|^s \cdot \left| 1 - \left| \frac{1 - \overline{z} \cdot e^{\lambda i}}{1 - z \cdot e^{\lambda i}} \right| \right| . \tag{12}$$

The following estimation can be easily seen by elementary arguments (or see in [3]): If z and w are non-real complex numbers with $zw \neq 1$, then there is a real number $c_7 > 0$ depending on z and $|w|$ such that

$$\left| 1 - \left| \frac{1 - \overline{z}w}{1 - zw} \right| \right| > \min\{1, c_7, |Im(w)|\}. \tag{13}$$

So by (11), (12) and (13)

$$\left| |\alpha|^s - \left| \frac{R_{n+s}}{R_n} \right| \right| > |\alpha|^s \cdot n^{-c_8} > n^{-c_9}$$

follow which proves inequality (6). □

Note. We note that we obtain similar results if the initial terms of the sequence $\{R_n\}$ are arbitrary, but in this case the constants are weaker and the expressions are more difficult.

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