

MIKUSIŃSKI FUNCTIONAL EQUATION ON A HEXAGON

KÁROLY LAJKÓ

Dedicated to Professor Árpád Varcza on the occasion of his 60th birthday

ABSTRACT. The general solution of the conditional functional equation (M) is described for functions $f: (-r, r) \rightarrow \mathbb{R}$, where (M) is satisfied for all $(x, y) \in H$, where $H = \{(x, y) \mid x, y, x + y \in (-r, r)\}$ is a hexagon.

1. INTRODUCTION

J. Mikusiński (in 1971) mentioned the functional equation

$$(M) \quad f(x+y)[f(x+y) - f(x) - f(y)] = 0$$

which since has been named after him.

The authors of [2] find the general solution of (M) for functions $f: X \rightarrow Y$ where $(X, +)$ and $(Y, +)$ are (not necessarily commutative) groups. In case $X = Y = \mathbb{R}$ they proved the following

Theorem 1. *The only solutions of equation (M) for functions $f: \mathbb{R} \rightarrow \mathbb{R}$ are additive functions, i.e. the solutions of Cauchy functional equation*

$$(1) \quad f(x+y) = f(x) + f(y) \quad (x, y \in \mathbb{R}).$$

The aim of this paper is to present the general solution of (M) for functions $f: (-r, r) \rightarrow \mathbb{R}$, where (M) is satisfied for all $(x, y) \in H = \{(x, y) \mid x, y, x + y \in (-r, r)\}$ and $(-r, r)$ is an open interval in $\mathbb{R}$.

2. AN EXTENSION THEOREM FOR (M)

Following the ideas of ACZÉL [1] and KUCZMA [3] we prove the following extension theorem for the Mikusiński functional equation (M).

Theorem 2. *If the function $f: (-r, r) \rightarrow \mathbb{R}$ satisfies the Mikusiński functional equation (M) for all $(x, y) \in H$, where H is a hexagon given above, then there exists a unique function $F: \mathbb{R} \rightarrow \mathbb{R}$ satisfying (M) for any $x, y \in \mathbb{R}$ and*

$$f(x) = F(x), \quad x \in (-r, r).$$

Proof. a) First we show that

$$(2) \quad f\left(\frac{x}{2^n}\right) = \frac{1}{2^n}f(x), \quad x \in (-r, r), \quad n \in \mathbb{N}.$$

If $f(x) = 0$, then it is easy to see that $f(2^n x) = 0$ ($2^n x \in (-r, r)$), which implies (2).

2000 *Mathematics Subject Classification.* 39B22.

Key words and phrases. Functional equations, extension theorem, general solution.

Research supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. T-030082 and by the Hungarian High Educational Research and Development Found (FKFP) Grant No. 0310/1997.

If $f(x) \neq 0$, then replacing both x and y by $\frac{x}{2}$, we get from (M)

$$(3) \quad f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \neq 0, \quad x \in (-r, r).$$

Thus (2) holds for $n = 1$.

Using (3) repeatedly completes the statement.

b) On the other hand, for every $u \in \mathbb{R}$ there exists an $n \in N \cup \{0\}$ such that $x = \frac{u}{2^n} \in (-r, r)$. We define the function F by

$$(4) \quad F: \mathbb{R} \rightarrow \mathbb{R}, \quad F(u) = 2^n f\left(\frac{u}{2^n}\right) \quad \left(\frac{u}{2^n} \in (-r, r)\right).$$

This definition is correct and (4) gives

$$f(x) = F(x), \quad x \in (-r, r).$$

c) We must verify that F satisfies (M) for all $x, y \in \mathbb{R}$.

If $x, y \in \mathbb{R}$ are arbitrary, then there exists an $n \in N \cup \{0\}$ such that

$$\frac{x}{2^n}, \frac{y}{2^n}, \frac{x+y}{2^n} \in (-r, r).$$

Now

$$f\left(\frac{x}{2^n} + \frac{y}{2^n}\right) \left[f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) - f\left(\frac{x}{2^n} + \frac{y}{2^n}\right) \right] = 0,$$

that is

$$2^n f\left(\frac{x+y}{2^n}\right) \left[2^n f\left(\frac{x}{2^n}\right) + 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{x+y}{2^n}\right) \right] = 0.$$

This implies that the function F , defined by (4) satisfies (M) for all $x, y \in \mathbb{R}$.

d) To prove the uniqueness, suppose that a function $G: \mathbb{R} \rightarrow \mathbb{R}$ satisfies (M) in $\mathbb{R}$ and fulfills the condition

$$(5) \quad G(x) = f(x), \quad x \in (-r, r).$$

Similarly as in a) one can get that

$$(6) \quad G\left(\frac{x}{2^n}\right) = \frac{1}{2^n}G(x), \quad x \in \mathbb{R}, \quad n \in N \cup \{0\}.$$

Take an arbitrary $x \in \mathbb{R}$. There exists an $n \in N \cup \{0\}$ such that $\frac{x}{2^n} \in (-r, r)$.

Thus we have by (4), (5) and (6)

$$G(x) = 2^n G\left(\frac{x}{2^n}\right) = 2^n f\left(\frac{x}{2^n}\right) = F(x).$$

Consequently $G = F$ in $\mathbb{R}$. □

3. THE GENERAL SOLUTION OF (M) ON A HEXAGON

Using Theorems 1 and 2 we obtain

Theorem 3. *If the function $f: (-r, r) \rightarrow \mathbb{R}$ satisfies the Mikusiński functional equation (M) for all $(x, y) \in H$, then there exists a unique additive function*

$A: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(7) \quad f(x) = A(x), \quad x \in (-r, r).$$

Proof. Theorem 2 shows that there exists a unique function $F: \mathbb{R} \rightarrow \mathbb{R}$ satisfying (M) for all $x, y \in \mathbb{R}$ and $f(x) = F(x)$ $x \in (-r, r)$.

Because of Theorem 1 F is an additive function.

It is easy to see that all additive functions A fulfill also (M) for all $(x, y) \in H$. □

REFERENCES

- [1] ACZÉL, J., Diamonds are not the Cauchy extensionists' best friend *Compt. Rendus Math. Dep. Acad. Sci. Canada* **5** no. 6 (1983), 259–264 .
- [2] DUBIKAJTIS, L. - FERENS, C. - GER, R. - KUCZMA, M., On Mikusiński's functional equation *Ann. Pol. Math.* **28** (1973), 40–47.
- [3] KUCZMA, M., *An Introduction to the Theory of Functional Equations and Inequalities* , Państwowe Wydawnictwo Naukowe Uniw. Śląski, Warszawa – Kraków – Katowice 1985.

Received December 1, 2000.

INSTITUTE OF MATHEMATICS AND INFORMATICS,
UNIVERSITY OF DEBRECEN,
H-4010 DEBRECEN, Pf. 12, HUNGARY
E-mail address: `lajko@math.klte.hu`