

DISCRETE LAGUERRE FUNCTIONS AND EQUILIBRIUM CONDITIONS

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Dedicated to the 60th birthday of Professor Árpád Varcza

ABSTRACT. The discrete Laguerre functions L_n^a ($n \in \mathbb{N}$) forms an orthonormal system on the unite circle $\mathbb{T}$ and the finite set of functions L_n^a ($n = 0, 1, \dots, N-1$) is orthonormal with respect to a discrete scalar product defined by the discrete subset $\mathbb{T}_N^a$ of $\mathbb{T}$. It is showed that the set $\mathbb{T}_N^a$ can be interpreted as a solution of an electrostatic equilibrium problem.

1. INTRODUCTION

In the control theory the discrete Laguerre functions and their generalizations are often used to identify the transfer function of the system [1], [2], [3]. The discrete Laguerre functions L_n^a ($n \in \mathbb{N}$) contain a complex parameter $a \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and can be expressed by the Blaschke functions

$$B_a(z) := \frac{z - a}{1 - \bar{a}z} \quad (z \in \mathbb{C}).$$

Namely (see [2])

$$L_n^a(z) := \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z} B_a^n(z) = \sqrt{1 - |a|^2} \frac{(z - a)^n}{(1 - \bar{a}z)^{n+1}} \quad (z \in \mathbb{C}).$$

The Laguerre functions form an orthonormal system on the unite circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, i.e.

$$\langle L_n^a, L_m^a \rangle := \frac{1}{2\pi} \int_0^{2\pi} L_n^a(e^{it}) \overline{L_m^a(e^{it})} dt = \delta_{mn} \quad (m, n \in \mathbb{N}),$$

where δ_{mn} is the Kronecker symbol (see [2]). A generalization of this system is the Malmquist–Takenaka system (see [4]).

If a belongs to $\mathbb{D}$ then B_a is a $1-1$ map on $\mathbb{D}$ and on $\mathbb{T}$, respectively. Moreover (see [2]) B_a can be written in the form

$$(1.1) \quad B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R}, a = re^{i\alpha} \in \mathbb{D}),$$

where

$$\beta_a(t) := \alpha + \gamma_s(t - \alpha), \quad \gamma_s(t) := 2 \arctan(s \tan \frac{t}{2}) \quad (t \in [-\pi, \pi)), \quad s := \frac{1+r}{1-r}$$

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and γ_s is extended to $\mathbb{R}$ by $\gamma_s(t + 2\pi) = 2\pi + \gamma_s(t)$ ($t \in \mathbb{R}$). The finite collection of the functions L_n^a ($0 \leq n < N$) form a discrete orthonormal system with respect to the scalar product

$$[f, g]_N := \frac{1}{N} \sum_{z \in \mathbb{T}_N^a} f(z) \overline{g(z)} \rho_a(z),$$

where

$$\mathbb{T}_N^a := \left\{ e^{i\beta_a^{-1}(t)} : t = \frac{2k\pi}{N}, k = 0, 1, \dots, N-1 \right\},$$

$$\rho_a(z) := \frac{|1 - \bar{a}z|^2}{1 - |a|^2} \quad (z \in \mathbb{T}) \quad \text{and} \quad \beta_a^{-1}(t) = \alpha + \gamma_{s-1}(t - \alpha) \quad (t \in \mathbb{R})$$

is the inverse function of β_a (see [2]). Namely, it is easy to check that

$$[L_n^a, L_m^a] = \delta_{mn} \quad (0 \leq m, n < N).$$

Indeed, by the definition of L_n^a , $\mathbb{T}_N^a$ and (1.1) and by the orthogonality of the discrete trigonometric system, for $0 \leq m, n < N$ we have

$$\begin{aligned} [L_n^a, L_m^a]_N &:= \frac{1}{N} \sum_{z \in \mathbb{T}_N^a} L_n^a(z) \overline{L_m^a(z)} \rho_a(z) = \\ &= \frac{1}{N} \sum_{z \in \mathbb{T}_N^a} B_a^{n-m}(z) = \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i(n-m)k/N} = \delta_{mn}. \end{aligned}$$

In this paper we show that the points of $\mathbb{T}_N^a$ are the solution of the following problem of electrostatic equilibrium:

Problem. Let $a \in \mathbb{D}$ and $N \in \mathbb{N}$, $N \geq 2$ be given numbers. If N unite “masses” at the variable point $z_1, z_2, \dots, z_N \in \mathbb{T}$ and two fixed masses $q = -(N-1)/2$ at a and $\bar{a}^{-1}$ are considered, for what position of the points $z_1, z_2, \dots, z_N$ does the potential function

$$(1.2) \quad V(z_1, \dots, z_N) := -\log \left(\prod_{1 \leq i < j \leq N} |z_i - z_j| \prod_{1 \leq i \leq N} |z_i - a|^q |z_i - \bar{a}^{-1}|^q \right) \\ (z_1, \dots, z_N \in \mathbb{T})$$

become a minimum ?

The function V can be interpreted as the energy of the electrostatic masses just defined. The minimum position corresponds to the condition of electrostatic equilibrium. Namely we have the following

Theorem. The numbers $w_k := e^{i\tau_k}$, $\tau_k := \beta_a^{-1}(2\pi(k-1)/N)$ ($k = 1, \dots, N$) of $\mathbb{T}_N^a$ are the solution of the equilibrium equations

$$(1.3) \quad \sum_{k=1, k \neq n}^N \frac{1}{w_n - w_k} + \frac{q}{w_n - a} + \frac{q}{w_n - \bar{a}^{-1}} = 0 \quad (n = 1, 2, \dots, N).$$

Moreover the point $(\tau_1, \tau_2, \dots, \tau_N) \in \mathbb{R}^N$ is a stationary point of the potential

$$V(e^{i\tau_1}, \dots, e^{i\tau_N}) \quad ((t_1, \dots, t_N) \in \mathbb{R}^N),$$

i.e.

$$(1.4) \quad \frac{\partial V(e^{i\tau_1}, \dots, e^{i\tau_N})}{\partial t_n} = 0 \quad (n = 1, \dots, N).$$

The system of equations (1.3) can be interpreted as the electrostatic equilibrium condition for the masses in question. We remark that *the zeros of the Jacobi, Laguerre and Hermite polynomials admit a similar interpretation* (see [6], pp. 140, 153).

A generalization of these results with respect to the Malmquist–Takenaka systems will be published in [5].

2. THE EQUILIBRIUM CONDITION

First we show (1.3).

Proof of (1.3). Denote

$$\varphi(z) := B_a^N(z) - 1 = \frac{(z-a)^N - (1-\bar{a}z)^N}{(1-\bar{a}z)^N} \quad (z \in \mathbb{C}).$$

By (1.1) it is clear that $\varphi(z) = 0$ if and only if $z = w_k := e^{i\beta_a^{-1}(2\pi k/N)}$ ($k = 1, 2, \dots, N$). Set

$$f(z) := \prod_{k=1}^N (z - w_k) \quad (z \in \mathbb{C}).$$

Since the polynomials f and $g(z) := (z-a)^N - (1-\bar{a}z)^N$ ($z \in \mathbb{C}$) have the same degree and roots, therefore $f(z) = \lambda[(z-a)^N - (1-\bar{a}z)^N]$ ($z \in \mathbb{C}$) with a constant $\lambda \in \mathbb{C}$.

It is easy to see that

$$(2.1) \quad \frac{1}{2} \frac{g''(w_n)}{g'(w_n)} = \frac{1}{2} \frac{f''(w_n)}{f'(w_n)} = \sum_{k=1}^N \frac{1}{w_n - w_k} \quad (n = 1, 2, \dots, N).$$

On the other hand by the definition of the function g and by

$$\left(\frac{w_n - a}{1 - \bar{a}w_n} \right)^N = 1 \quad (n = 1, 2, \dots, N)$$

we have

$$\begin{aligned} \frac{g''(w_n)}{g'(w_n)} &= \frac{N(N-1)(w_n-a)^{N-2} - \bar{a}^2 N(N-1)(1-\bar{a}w_n)^{N-2}}{N(w_n-a)^{N-1} + \bar{a}N(1-\bar{a}w_n)^{N-1}} = \\ &= \frac{\frac{N(N-1)}{(w_n-a)^2} - \frac{\bar{a}^2 N(N-1)}{(1-\bar{a}w_n)^2}}{\frac{N}{w_n-a} + \frac{\bar{a}N}{1-\bar{a}w_n}}. \end{aligned}$$

Hence we get

$$\begin{aligned} (2.2) \quad \frac{g''(w_n)}{g'(w_n)} &= (N-1) \frac{\frac{1}{(w_n-a)^2} - \frac{1}{(w_n-\bar{a}^{-1})^2}}{\frac{1}{w_n-a} - \frac{1}{w_n-\bar{a}^{-1}}} = \\ &= (N-1) \left(\frac{1}{w_n-a} + \frac{1}{w_n-\bar{a}^{-1}} \right). \end{aligned}$$

Comparing (2.1) and (2.2) we get (1.3). $\square$

Now we show that for the solution of (1.3) the condition (1.4) is satisfied.

Proof of (1.4). Set $a := re^{i\alpha}$. Then by (1.3) we have

$$\begin{aligned} \sum_{k=1}^N \frac{\cos \tau_n - \cos \tau_k}{|w_n - w_k|^2} + \frac{q(\cos \tau_n - r \cos \alpha)}{|w_n - a|^2} + \frac{q(\cos \tau_n - r^{-1} \cos \alpha)}{|w_n - \bar{a}^{-1}|^2} &= 0, \\ \sum_{k=1}^N \frac{\sin \tau_n - \sin \tau_k}{|w_n - w_k|^2} + \frac{q(\sin \tau_n - r \sin \alpha)}{|w_n - a|^2} + \frac{q(\sin \tau_n - r^{-1} \sin \alpha)}{|w_n - \bar{a}^{-1}|^2} &= 0 \\ &\quad (n = 1, 2, \dots, N). \end{aligned}$$

Multiplying the first equality by $\sin \tau_n$ the second by $\cos \tau_n$ and taking the difference we get

$$(2.3) \quad \sum_{k=1}^N \frac{\sin(\tau_n - \tau_k)}{|w_n - w_k|^2} + \frac{qr \sin(\tau_n - \alpha)}{|w_n - a|^2} + \frac{qr^{-1} \sin(\tau_n - \alpha)}{|w_n - \bar{a}^{-1}|^2} = 0 \quad (k = 1, \dots, N).$$

By the definition of V

$$\begin{aligned} V(e^{it_1}, \dots, e^{it_N}) &= -\log \left(\prod_{1 \leq j < k \leq N} |e^{it_j} - e^{it_k}| \prod_{1 \leq j \leq N} |e^{it_j} - a|^q |e^{it_j} - \bar{a}^{-1}|^q \right) = \\ &= -\log \left(\prod_{1 \leq j < k \leq N} |e^{i(t_j - t_k)} - 1| \prod_{1 \leq j \leq N} |e^{i(t_j - \alpha)} - r|^q |e^{i(t_j - \alpha)} - r^{-1}|^q \right) \\ &\quad (0 \leq t_1 \leq \dots \leq t_N < 2\pi). \end{aligned}$$

It is easy to check that

$$\begin{aligned} \frac{\partial V(e^{i\tau_1}, \dots, e^{i\tau_N})}{\partial t_n} &= \\ &= \sum_{k=1}^N \frac{\sin(\tau_n - \tau_k)}{|w_n - w_k|^2} + \frac{qr \sin(\tau_n - \alpha)}{|w_n - a|^2} + \frac{qr^{-1} \sin(\tau_n - \alpha)}{|w_n - \bar{a}^{-1}|^2} \\ &\quad (k = 1, \dots, N). \end{aligned}$$

and by (2.3) we get (1.4). $\square$

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