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ON THE RECTIFIABILITY CONDITION OF A SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

SÁNDOR BÁCSÓ, ÁGOTA OROSZ AND BRIGITTA SZILÁGYI

Dedicated to Professor Árpád Varecza on his 60th birthday

Abstract. In this paper we wish to survey the rectifiability conditions of a second order differential equation, and we give some examples for a projectively flat two-dimensional Finsler space.

1. Introduction

In a famous book of Arnold [2] we can find the following theorem: "An equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$ can be reduced to the form $d^2\overline{y}/d\overline{x}^2 = 0$ if and only if the right-hand side is a polynomial in the derivative of order not greater 3 both for the equation and for its dual."

This theorem can be formulated in the following form on the basis of [4]: "An equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$ can be reduced to the form $d^2\overline{y}/d\overline{x}^2 = 0$ if and only if the path space P^2 (determined by the equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$) is projectively related to a two-dimensional projectively flat Finsler space F^2."

The aim of this paper is to give some projectively flat two-dimensional Finsler spaces using this latter theorem.

2. Notations and theorems

Proposition 2.1. [1] *The second order differential equations*

$$d^2x^i/dt^2 = -2G^i(x, \dot{x});\ \dot{x}^i = dx^i/dt\ (i = 1, 2, \dots, n)$$

give a path space P^n, where the functions $G^i(x, \dot{x})$ are positively homogeneous of degree two in $\dot{x}$.

Definition 2.2. [2] The integral curves of this second order differential equation are called paths.

Definition 2.3. [1] A Finsler space F^n is a pair (M^n, L), where M^n is a connected differentiable manifold of dimension n, $L(x, \dot{x})$ is the metrical function defined on the manifold TM/O of nonzero tangent vectors, and $L(x, \dot{x})$ is positively homogeneous of degree one in $\dot{x}$.

The differential equations of geodesic curves of F^n:

$$d^2x^i/dt^2 = -2G^i(x, \dot{x})\ ,$$

where

$$G^i = g^{ij}\left[\dot{x}^r \partial^2 L^2/\partial\dot{x}^j \partial x^r - \partial L^2/\partial x^j\right]\ ,$$

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and

$$g^{ij} = (g_{ij})^{-1} \quad ; \quad g_{ij} = \frac{1}{2}\partial^2 L^2/\partial\dot{x}^i\partial\dot{x}^j .$$

Definition 2.4. [1]A path space P^n and a Finsler space F^n are called projectively related to each other, if any path of P^n is a geodesic curve of F^n and vice versa.

Theorem 2.5. [5] *In two-dimensions any path space is projectively related to a two dimensional Finsler space.*

Definition 2.6. [1] A Finsler space is called projectively flat, if it is covered by coordinate neighborhoods in which any geodesic is represented by linear equations.

Theorem 2.7. [1] *If a path space P^n projectively is related to a Finsler space F^n, then we have two invariant tensors, called the Weyl and the Douglas tensor respectively.*

Definition 2.8. [3] A Finsler space is said to be a Douglas space, if the Douglas tensor D^h_{ijk} vanishes identically, where

$$D^h_{ijk} = \partial^3 Q^h/\partial\dot{x}^k\partial\dot{x}^j\partial\dot{x}^i$$

with $Q^h = G^h - \dot{x}^h G\dfrac{1}{n+1} \quad ; \quad G = G^r_r \quad ; \quad G^i_j = \partial G^i/\partial\dot{x}^j$.

Theorem 2.9. [3] *A two-dimensional Finsler space F^2 is a Douglas space if and only if (in a local coordinate system (x, y)) the right-hand side $\Phi(x, y, dy/dx)$ of the equation of geodesics $y'' = \Phi(x, y, dy/dx)$ is a polynomial in $dy/dx = y'$ of degree at most three.*

From the previous Theorems and Definitions we obtain

THEOREM 1. *An equation $d^2y/dx^2 = \Phi(x, y, y')$ can be reduced to the form $d^2\overline{y}/d\overline{x}^2 = 0$ if the pathspace P^2 (determined by the equation $d^2y/dx^2 = \Phi(x, y, y')$) is projective related to a two-dimensional Douglas space.*

Some examples can be found in the papers [3] and [4] for the Douglas spaces.

Theorem 2.10. [4] *A Finsler space F^2 (a two-dimensional Finsler space) is a projectively flat space if and only if F^2 is a Douglas space and satisfies $\Pi_{ijk} = 0$, where $\Pi_{ijk} = \partial Q_{ij}/\partial x^k + Q^r_{ij}Q_{rk} - [ij]$. The tensor $Q_{ij} = Q^r_{ijr}$, where in a Douglas space*

$$Q^h_{ijk} = \partial Q^h_{ij}/\partial x^k + Q^r_{ij}Q^h_{rk} - [ij] \quad , \quad Q^h_{ij} = \frac{\partial^2\left[G^h - \dot{x}^h G\frac{1}{n+1}\right]}{\partial\dot{x}^i\partial\dot{x}^j} .$$

Theorem 2.11. [4] *A two-dimensional Finsler space F^2 is a Douglas space if and only if the differential equation of F^2 has the form*

$$\begin{aligned} y'' &= k(x,y)(y')^3 + h(x,y)(y')^2 + g(x,y)y' + f(x,y) = \\ &= Q^1_{22}(y')^3 - (Q^2_{22} - 2Q^1_{12})(y')^2 - (2Q^2_{12} - Q^1_{11})y' + Q^2_{11} . \end{aligned}$$

Theorem 2.12. [4] *A two dimensional Douglas space is projectively flat if and only if $\Pi_{112} = 0$ and $\Pi_{212} = 0$.*

THEOREM 2. *In a Douglas space*

$$\Pi_{112} = -f_{yy} + \frac{2}{3}g_{xy} - \frac{1}{3}gg_y + fh_y + hf_y - \frac{1}{3}h_{xx} + \frac{1}{3}h_x g + kf_x - \frac{2}{27}g^2h + \frac{2}{3}gkh - \frac{2}{3}fhk ,$$

$$\begin{aligned} \Pi_{212} = &- \frac{1}{3}g_{yy} + \frac{2}{3}h_{xy} - \frac{1}{3}g_y h + k_y f + 2kf_y - k_{xx} + \frac{2}{3}hh_x + \frac{2}{3}h_x k - \frac{4}{27}gh^2 + \\ &+ \frac{2}{3}hk_x - \frac{1}{3}gk_x - \frac{1}{3}g_x k - \frac{2}{9}hgk + \frac{2}{9}g^2k , \end{aligned}$$

where $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, ...

3. Some examples

Assume that a two-dimensional Finsler space F^2 on a domain of the (x, y)-plane has the geodesics given by the equations:

1. $y'' = f(x, y)$,
2. $y'' = g(x, y)y' + f(x, y)$,
3. $y'' = k(x, y)(y')^3$,
4. $y'' = h(x, y)(y')^2$,
5. $y'' = g(x, y)y'$.

The components of the tensor Π are the following in these cases:

1. $\Pi_{112} = -f_{yy}$; $\Pi_{212} = 0$,
2. $\Pi_{112} = -f_{yy} + \frac{2}{3}g_{xy} - \frac{1}{3}gg_y$; $\Pi_{212} = -\frac{1}{3}g_{yy}$,
3. $\Pi_{112} = 0$; $\Pi_{212} = -k_{xx}$,
4. $\Pi_{112} = -\frac{1}{3}h_{xx}$; $\Pi_{212} = h_{xy} + hh_x$,
5. $\Pi_{112} = \frac{2}{3}g_{xy} - \frac{1}{3}gg_y$; $\Pi_{212} = -\frac{1}{3}g_{yy}$.

Consequently, F^2 is projectively flat, if and only if

1. $f(x, y) = A(x)y + B(x)$,
2. $f(x, y) = \sigma_1(x)y^3 + \sigma_2(x)y^2 + \sigma_3(x)y + \sigma_4(x)$; $g(x, y) = \alpha(x)y + \beta(x)$,
3. $k(x, y) = C(y)x + D(y)$,
4. $h(x, y) = E(y)x + F(y)$, where $dE/dy + (Ex + F)E = 0$,
5. $g(x, y) = \gamma(x)y + \delta(x)$, where $\frac{2}{3}dy/dx - \frac{1}{3}(\gamma y + \delta)\gamma = 0$.

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Institute of Mathematics and Informatics,
University of Debrecen,
H-4010 Debrecen, Pf. 12, Hungary
E-mail address: bacsos@math.klte.hu
E-mail address: oagota@math.klte.hu