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ON THE CHERN–WEIL HOMOMORPHISM IN FINSLER SPACES

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Dedicated to Professor Árpád Varecza on the occasion of his 60th birthday

Abstract. The aim of this paper is to devise a Chern – Weil-type construction for a Finsler manifold $(M, \mathcal{L})$ which is determined only by the manifold M and by the Finslerian fundamental function $\mathcal{L}$.

1. Introduction

The focus in this paper is to set up a framework in which the famous Chern–Weil homomorphism can be formulated on a Finsler manifold. Most of the basic notations in this paper are the same as in [GHV73]. Background information on Finsler geometry can be found e.g. in [Mat86] and [AP94].

Let $(M, \mathcal{L})$ be a Finsler space, the horizontal projection determined by the Finslerian fundamental function $\mathcal{L}$ is h. h can be interpreted as a τ_{TM}-valued 1-form on TM: $h \in \mathrm{A}^1(TM; \tau_{TM}) \cong \mathrm{Hom}(\tau_{TM}; \tau_{TM})$. The horizontal subbundle of τ_{TM} will be denoted by HM, $\mathrm{Sec}\, HM = \mathfrak{X}_h(TM)$.

Denote by $(\mathrm{A}(TM), \wedge)$ the graded algebra of differential forms on TM. From h one can derive a first order graded derivative $d_h \colon \mathrm{A}(TM) \to \mathrm{A}(TM)$

$$\begin{aligned} d_h\omega(X_0, \dots, X_p) &= \\ &= \sum_{i=0}^{p} (-1)^i hX_i \omega(X_0, \dots, \widehat{X}_i, \dots, X_p) + \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega\left([X_i, X_j]_h, X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_p \right) \end{aligned}$$

where $\omega \in \mathrm{A}^p(TM)$ $(p \geq 1)$ is a p-form, $X_i \in \mathfrak{X}(TM)$ $(i = 0 \dots p)$, $[X, Y]_h = [hX, Y] + [X, hY] - h[X, Y]$, furthermore $d_h f(X) = (hX)f$ $(f \in \mathrm{A}^0(TM) \equiv C^\infty(TM))$ ([FN56] or [Mic87]). It is easy to see that $d_h^2 = 0$ iff the Frölicher–Nijenhuis bracket of the operator pair (h, h) is zero: $[h, h] = 0$. In the Finslerian case this condition means that the torsion R^1 of the unique Cartan connection vanishes, i.e. the horizontal distribution is integrable.

In the Finslerian case this special situation was studied in [ACD87] and their main result is the following:

Theorem. *If $R^1 = 0$ then the cohomology groups of d_h are isomorphic to the de Rham cohomology groups of an integral manifold of the nonlinear connection associated to $\mathcal{L}$.*

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In this paper we do not suppose the integrability of the horizontal distribution.

2. Tools

Forms. Let $\nabla^C\colon \mathfrak{X}(TM)\times\mathfrak{X}_hTM\to\mathfrak{X}_hTM$ be the Cartan connection of the Finsler space $(M,\mathcal{L})$. Then (∇,h), where $\nabla\colon \mathfrak{X}(TM)\times\mathfrak{X}_hTM\to\mathfrak{X}_hTM$, $\nabla_XY={\nabla^C}_{hX}Y$, is the so called *h-connection* of the Finsler space.

By easy calculations, one can show the following statement:

Proposition 1. *Let (∇,h) be the h-connection of the Finsler space. The map*

$$\nabla\colon A(TM;HM)\to A(TM;HM)\,,$$

$$(\nabla\Psi)(X_0,\dots,X_p)=\sum_{i=0}^{p}(-1)^i\nabla_{X_i}\Psi(X_0,\dots,\widehat{X}_i,\dots,X_p)+ \\ +\sum_{i<j}(-1)^{i+j}\Psi([X_i,X_j]_h,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_p)$$

($\Psi\in A^p(TM;HM)$ $(p>0)$; for $p=0:$ $(\nabla\sigma)(X)=\nabla_X\sigma$) is a first order graded derivation of the graded algebra of HM-valued forms on TM in the sense of [Mic87] *i.e.* $\nabla(\omega\wedge\Psi)=d_h\omega\wedge\Psi+(-1)^p\omega\wedge\nabla\Psi;\ \omega\in A^p(TM)\,,\ \Psi\in A(TM;HM)\,.$

We will use the following construction in the next section. Let $\xi_0,\xi_1,\dots,\xi_m$ be vector bundles with the same base B and let $\phi\in\mathrm{Hom}(\xi_1,\dots,\xi_m;\xi_0)$. ϕ determines a map $\phi_*\in\mathrm{Hom}(\mathrm{A}(B;\xi_1)\,,\dots,\mathrm{A}(B;\xi_m)\,;\mathrm{A}(B;\xi_0))$ as follows.

$$\phi_*(\sigma_1,\dots,\sigma_m)=\phi(\sigma_1,\dots,\sigma_m)$$

for $\sigma_i\in\mathrm{A}^0(B;\xi_i)\equiv\mathrm{Sec}\xi_i$ and for elements in $\mathrm{A}^{p_i}(B;\xi_i)$ this map is determined by

$$\phi_*(\omega_1\wedge\sigma_1,\dots,\omega_m\wedge\sigma_m)=(\omega_1\wedge\dots\wedge\omega_m)\wedge\phi_*(\sigma_1,\dots,\sigma_m)$$

where $\omega_i\in\mathrm{A}(B)\,,\ \sigma_i\in\mathrm{A}^0(B;\xi_i)\ \ (i=1\dots m)$. ϕ_* satisfies the following identity:

$$\phi_*(\Psi_1,\dots,\omega\wedge\Psi_i,\dots,\Psi_m)=(-1)^{q_iq}\omega\wedge\phi_*(\Psi_1,\dots,\Psi_m) \tag{1}$$

where $\Psi_i\in\mathrm{A}^{p_i}(B;\xi_i)\,,\ q_i=p_1+\dots+p_{i-1}\ (i\geq 2),\ q_1=0,\ \omega\in\mathrm{A}^q(B)$, and moreover,

$$\begin{aligned}&\phi_*(\Psi_1,\dots,\Psi_m)(X_1,\dots,X_p)=\\ &=\frac{1}{p_1!\cdots p_m!}\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)\phi(\Psi_1(X_{\sigma(1)},\dots),\dots,\Psi_m(\dots,X_{\sigma(p)}))\end{aligned} \tag{2}$$

where $\Psi_i\in\mathrm{A}^{p_i}(B;\xi_i)\,,\ X_i\in\mathfrak{X}(B)\ (i=1\dots p),\ p=p_1+\dots+p_m$.

Invariant polynomials. Let F be a real vector space. An *invariant polynomial of degree p* is a symmetric map

$$f_F^p\in\mathrm{Hom}(\mathrm{L}(\overset{1}{\smile}\!\!\!\!F;F),\dots,\mathrm{L}(\overset{p}{\smile}\!\!\!\!F;F);\mathbb{R})$$

such that for all $a\in\mathrm{GL}(F)$

$$f_F^p(\mathrm{Ad}\,(a)\alpha_1,\dots,\mathrm{Ad}\,(a)\alpha_p)\quad=\quad f_F^p(\alpha_1,\dots,\alpha_p) \tag{3}$$

where $\alpha_i\in\mathrm{L}(F;F)$ $(i=1\dots p)$ is a linear operator and $\mathrm{Ad}:\mathrm{Gl}(F)\to\mathrm{Gl}(\mathrm{L}(F;F))$ is the adjoint representation. By the invariance condition (3) one can extend f_F^p to the bundle of linear operators over the vector bundle ξ with base manifold B and the typical fiber F.

$$f^p\in\mathrm{Hom}(\overset{1}{\smile}\!\!\!\!\mathrm{L}_\xi,\dots,\overset{p}{\smile}\!\!\!\!\mathrm{L}_\xi;B\times\mathbb{R})\cong\mathrm{Sec}(\overset{1}{\smile}\!\!\!\!\mathrm{L}_\xi\otimes\dots\otimes\overset{p}{\smile}\!\!\!\!\mathrm{L}_\xi)^*.$$

This f^p is called invariant polynomial in ξ of degree p.

Curvature. Let $R^2 \in \mathrm{A}^2(TM; \mathrm{L}_{HM})$ denote the curvature of the Cartan connection.

Proposition 2. *The h-Finsler connection (∇, h) in HM satisfies:*

$$\left[\left(\nabla R^2\right)(X, Y, Z)\right](W) = \underset{(X,Y,Z)}{\mathfrak{S}} \left\{P^2(R^1(X, Y)(hZ)(W))\right\}$$

where P^2 is the hv-curvature, $R^1 = \frac{1}{2}[h, h]$ and $\mathfrak{S}_{(X,Y,Z)}$ is the symbol of the cyclic sum with respect to X, Y, Z.

3. Construction of d_h-closed forms

Theorem. *Let (∇, h) be the h-Finsler connection, f^p an invariant polynomial in HM. If $\nabla R^2 = 0$ then $d_h f^p_*(R^2, \ldots, R^2) = 0$, i.e. $f^p_*(R^2, \ldots, R^2)$ is a d_h-closed 2p-form.*

Proof. We have found the adequate ideas, so the proof of the theorem is quite easy. First we prove the following statement:

Lemma. *Let $f \in Hom(\overset{1}{\smile}{L_{HM}}, \ldots, \overset{p}{\smile}{L_{HM}}; TM \times \mathbb{R}) \cong Sec(\overset{1}{\smile}{L_{HM}} \otimes \cdots \otimes \overset{p}{\smile}{L_{HM}})^*$. If $\nabla_X f = 0$ for any $X \in \mathfrak{X}(TM)$ then*

$$d_h f_*(\Omega_1, \ldots, \Omega_p) = \sum_{i=1}^{p} (-1)^{q_i} f_*(\Omega_1, \ldots, \nabla\Omega_i, \ldots, \Omega_p), \tag{4}$$

where $\Omega_i \in A^{r_i}(TM; L_{HM})$ $(i = 1 \ldots p)$, $q_i = r_1 + \cdots + r_{i-1}$ $(i = 2 \ldots p)$, $q_1 = 0$. (Concerning $f_* \in Hom(A(\overset{1}{\smile}{TM}; L_{HM}), \ldots, A(\overset{p}{\smile}{TM}; L_{HM}); A(TM))$ see (2)!)

Clearly, $\mathrm{A}^{r_i}(TM; \mathrm{L}_{HM}) \cong \mathrm{A}^{r_i}(TM) \otimes \mathrm{Sec}\mathrm{L}_{HM}$. If $\alpha_i \in \mathrm{Sec}\mathrm{L}_{HM}$ $(i = 1 \ldots p)$ then (4) reduces to:

$$d_h f(\alpha_1, \ldots, \alpha_p) = \sum_{i=1}^{p} f_*(\alpha_1, \ldots, \nabla\alpha_i, \ldots, \alpha_p).$$

We have $(d_h f(\alpha_1, \ldots, \alpha_p))(X) = hXf(\alpha_1, \ldots, \alpha_p)$. On the other hand,

$$\sum_{i=1}^{p} f_*(\alpha_1, \ldots, \nabla\alpha_i, \ldots, \alpha_p)(X) \overset{(2)}{=} \sum_{i=1}^{p} f(\alpha_1, \ldots, (\nabla\alpha_i)(X), \ldots, \alpha_p).$$

Together with the previous line this proves the statement.

Let $\Omega_i \in \mathrm{A}^{r_i}(TM; \mathrm{L}_{HM})$ $(i = 1 \ldots p)$, $\Omega_i = \omega_i \wedge \alpha_i$ $(\omega_i \in \mathrm{A}^{r_i}(TM)$ $i = 1 \ldots p)$, and $q = r_1 + \cdots + r_p$. By induction we infer

$$\begin{aligned} & d_h f_*(\omega_1 \wedge \alpha_1, \ldots, \omega_p \wedge \alpha_p) \overset{(1)}{=} \\ = \quad & \sum_{i=1}^{p} (-1)^{q_i} \omega_1 \wedge \ldots \wedge d_h\omega_i \wedge \ldots \wedge \omega_p \wedge f_*(\alpha_1, \ldots, \alpha_p) + \\ & + (-1)^q \omega_1 \wedge \ldots \wedge \omega_p \wedge d_h f_*(\alpha_1, \ldots, \alpha_p). \end{aligned}$$

Similarly,

$$\begin{aligned} & f_*(\omega_1 \wedge \alpha_1, \ldots, \nabla(\omega_i \wedge \alpha_i), \ldots, \omega_p \wedge \alpha_p) = \\ = \quad & \omega_1 \wedge \ldots \wedge d_h\omega_i \wedge \ldots \wedge \omega_p \wedge f_*(\alpha_1, \ldots, \alpha_p) + \\ & + (-1)^{r_i}(-1)^{r_{i+1}} \cdots (-1)^{r_p} \omega_1 \wedge \ldots \wedge \omega_p \wedge f_*(\alpha_1, \ldots, \nabla\alpha_i, \ldots, \alpha_p). \end{aligned}$$

We proved the lemma.

Now, for an invariant polynomial f^p, $\nabla_X f^p = \nabla^C{}_{hX} f^p = 0$ and applying the lemma for $\Omega_i = R^2$ we get the statement of the theorem. □

4. Remarks

Pseudocomplexes. For d_h we have a sequence of graded vector spaces

$$(\mathsf{PS}) \qquad \cdots \longrightarrow \mathrm{A}^{p-1}(TM) \xrightarrow{d_h} \mathrm{A}^{p}(TM) \xrightarrow{d_h} \mathrm{A}^{p+1}(TM) \longrightarrow \cdots$$

where $d_h \circ d_h$ is not necessarily zero. Following I. Vaisman [Vai68], for (PS) we use the name of *pseudocomplex*. Of course, when $[h, h] = 0$ then (PS) is a usual cochain complex.

In the case of non-vanishing d_h^2 the most natural way to define cohomology groups is by

$$H^p(d_h, TM) = {}^{\mathrm{Ker}\, d_h}\!\big/_{\mathrm{Im}\, d_h \cap \mathrm{Ker}\, d_h}.$$

These $H^p(d_h, TM)$ cohomology groups are usual cohomology groups of several cochain complexes. We put

$$(\widetilde{\mathsf{PS}}) \qquad \cdots \longrightarrow \widetilde{\mathrm{A}^{p-1}(TM)} \xrightarrow{\tilde{d}_h} \widetilde{\mathrm{A}^{p}(TM)} \xrightarrow{\tilde{d}_h} \widetilde{\mathrm{A}^{p+1}(TM)} \longrightarrow \cdots$$

where

$$\widetilde{\mathrm{A}^{p}(TM)} = \mathrm{Ker}\, d_h \circ d_h.$$

and $\tilde{d}_h$ is the restriction of d_h to $\widetilde{\mathrm{A}^{p}(TM)}$. Then it is easy to check that in the case of $(\widetilde{\mathsf{PS}})$ $\tilde{d}_h^2 = 0$ holds and the cochain complex $(\widetilde{\mathsf{PS}})$ has the same cocycles and coboundaries as the pseudocomplex (PS) itself ([HL75], [Vai93]).

Finsler spaces with the condition $\nabla R^2 = 0$. There are several examples for Finsler spaces with vanishing curvature P^2. This condition implies the required identity $\nabla R^2 = 0$, c.f. Proposition 2. These spaces are the so called *Landsberg* spaces ([Koz96], [Mat96]).

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