

WREATH PRODUCTS IN THE UNIT GROUP OF MODULAR GROUP ALGEBRAS OF 2-GROUPS OF MAXIMAL CLASS

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ABSTRACT. We study the unit group of the modular group algebra KG , where G is a 2-group of maximal class. We prove that the unit group of KG possesses a section isomorphic to the wreath product of a group of order two with the commutator subgroup of the group G .

1. INTRODUCTION

Let p be a prime number, G be a finite p -group and K be a field of characteristic p . Denote by $\Delta = \Delta_K(G)$ the augmentation ideal of the modular group algebra KG . The group of normalized units $U(G) = U(KG)$ consists of all elements of the type $1 + x$, where $x \in \Delta$. Our further notation follows [20].

Define Lie-powers $KG^{[n]}$ and $KG^{(n)}$ in KG : $KG^{[n]}$ is two-sided ideal, generated by all (left-normed) Lie-products $[x_1, x_2, \dots, x_n], x_i \in KG$, and $KG^{(n)}$ is defined inductively: $KG^{(1)} = KG$, $KG^{(n+1)}$ is the associative ideal generated by $[KG^{(n)}, KG]$. Clearly, for every n $KG^{(n)} \supseteq KG^{[n]}$, but equality need not hold.

For modular group algebras of finite p -groups $KG^{(|G'|+1)} = 0$ [24]. Then in our case finite lower and upper Lie nilpotency indices are defined:

$$t_L(G) = \min\{n : KG^{[n]} = 0\}, \quad t^L(G) = \min\{n : KG^{(n)} = 0\}.$$

It is known that $t_L(G) = t^L(G)$ for group algebras over the field of characteristic zero [19], and for the case of characteristic $p > 3$ their coincidence was proved by A. Bhandari and I. B. S. Passi [3].

Consider the following normal series in $U(G)$:

$$U(G) = 1 + \Delta \supseteq 1 + \Delta(G') \supseteq 1 + \Delta^2(G') \supseteq 1 + \Delta^{t(G')}(G'),$$

where $t(G')$ is the nilpotency index of the augmentation ideal of KG' .

An obvious question is whether does exist a refinement for this normal series. There were two conjectures relevant to the question above.

The first one, as it was stated in [20], is attributed to A. A. Bovdi and consists in the equality $\text{cl}U(G) = t(G')$, i.e. this normal series doesn't have a refinement. In particular, C. Baginski [1] proved that $\text{cl}U(G) = p$ if $|G'| = p$ (in case of cyclic commutator subgroup $t(G') = |G'|$). A. Mann and A. Shalev proved that $\text{cl}U(G) \leq t(G')$ for groups of class two [16].

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The second conjecture was suggested by S. A. Jennings [13] in a more general context, and in our case it means that $\text{cl}U(G) = t_L(G) - 1$. Here N. Gupta and F. Levin proved inequality $\leq$ in [11].

The first conjecture was more attractive and challenging, since methods for the systematic computation of the nilpotency index of the augmentation ideal $t(G')$ were more known than such ones for the calculation of the lower Lie nilpotency index (for key facts see, for example, [12], [15], [18], [21], [24]).

Moreover, A. Shalev [20] proved that these two conjectures are incompatible in general case, although $t(G') = t_L(G) - 1$ for some particular families of groups, including 2-groups of maximal class. Later using computer Coleman managed to find counterexample to Bovdi's conjecture (cf. [25]), and the final effort in this direction was made by X. Du [10] in his proof of Jennings conjecture.

Study of the structure of the unit group of group algebra and its nilpotency class raised a number of questions of independent interest, in particular, about involving of different types of wreath products in the unit group (as a subgroup or as a section).

In [9] D. Coleman and D. Passman proved that for non-abelian finite p -group G a wreath product of two groups of order p is involved into $U(KG)$. Later this result was generalized by A. Bovdi in [4]. Among other related results it is worth to mention [16], [17], [23].

It is also an interesting question whether $U(KG)$ possesses a given wreath product as a subgroup or only as a section, i.e. as a factor-group of a certain subgroup of $U(KG)$. Baginski in [1] described all p -groups, for which $U(KG)$ does not contain a subgroup isomorphic to the wreath product of two groups of order p for the case of odd p , and the case of $p = 2$ was investigated in [7].

The question whether $U(G)$ possesses a section isomorphic to the wreath product of a cyclic group C_p of order p and the commutator subgroup of G was stated by A. Shalev in [20]. Since the nilpotency class of the wreath product $C_p \wr H$ is equal to $t(H) - 1$ - the nilpotency index of the augmentation ideal of KH [8], this question was very useful for the investigation of the first conjecture. In [22] positive answer was given by A. Shalev for the case of odd p and a cyclic commutator subgroup of G .

The present paper is aimed to extend the last result on 2-groups of maximal class, proving that if G is such a group then the unit group of KG possesses a section isomorphic to the wreath product of a group of order two with the commutator subgroup of the group G . We prove the following main result.

Theorem 1. *Let K be a field of characteristic two, G be a 2-group of maximal class. Then the wreath product $C_2 \wr G'$ of a cyclic group of order two and the commutator subgroup of G is involved in $U(KG)$.*

2. PRELIMINARIES

We consider 2-groups of maximal class, namely, the dihedral, semidihedral and generalized quaternion groups, which we denote by D_n, S_n and Q_n respectively. They are given by following representations [2]:

$$\begin{aligned} D_n &= \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\ S_n &= \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{-1+2^{n-2}} \rangle, \\ Q_n &= \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle, \end{aligned}$$

where $n \geq 3$ (We shall consider D_3 and S_3 as identical groups).

We may assume that K is a field of two elements, since in the case of an arbitrary field of characteristic two we may consider its simple subfield and corresponding subalgebra in KG , where G is one of the groups D_n, S_n or Q_n .

Denote for $x = \sum_{g \in G} \alpha_g \cdot g$, $\alpha_g \in K$ by $\text{Supp } x$ the set $\{g \in G \mid \alpha_g \neq 0\}$. Since K is a field of two elements, $x \in 1 + \Delta \Leftrightarrow |\text{Supp } x| = 2k + 1, k \in \mathbb{N}$.

Next, for every element g in G there exists unique representation in the form $a^i b^j$, where $0 \leq i < 2^{n-1}, 0 \leq j < 2$. Then for every $x \in KG$ there exists unique representation in the form $x = x_1 + x_2 b$, where $x_i = a^{n_1} + \dots + a^{n_{k_i}}$. We shall call x_1, x_2 components of x . Clearly, $x_1 + x_2 b = y_1 + y_2 b \Leftrightarrow x_i = y_i, i = 1, 2$.

The mapping $x \mapsto \bar{x} = b^{-1} x b$, which we shall call *conjugation*, is an automorphism of order 2 of the group algebra KG . An element z such that $z = \bar{z}$ will be called *self-conjugated*.

Using this notions, it is easy to obtain the rule of multiplication of elements from KG , which is formulated in the next lemma.

Lemma 1. *Let $f_1 + f_2 b, h_1 + h_2 b \in KG$. Then*

$$(f_1 + f_2 b)(h_1 + h_2 b) = (f_1 h_1 + f_2 \bar{h}_2 \alpha) + (f_2 \bar{h}_1 + f_1 h_2) b,$$

where $\alpha = 1$ for D_n and S_n , $\alpha = b^2$ for Q_n .

We proceed with a pair of technical results.

Lemma 2. *An element $z \in KG$ commute with $b \in G$ if and only if z is self-conjugated.*

Lemma 3. *If x and y are self-conjugated, then $xy = yx$.*

In the next lemma we find the inverse element for an element from $U(KG)$.

Lemma 4. *Let $f = f_1 + f_2 b \in U(KG)$. Then $f^{-1} = (\bar{f}_1 + f_2 b)R^{-1}$, where $R = f_1 \bar{f}_1 + f_2 \bar{f}_2 \alpha$, and $\alpha = 1$ for D_n, S_n , $\alpha = b^2$ for Q_n .*

Proof. Clearly, R is a self-conjugated element of $K\langle a \rangle$ of augmentation 1, hence R is a central unit in KG . Then the lemma follows since

$$(f_1 + f_2 b)(\bar{f}_1 + f_2 b) = (\bar{f}_1 + f_2 b)(f_1 + f_2 b) = R.$$

Now we formulate another technical lemma, which is easy to prove by straightforward calculations using previous lemma.

Lemma 5. *Let $f, h \in U(G)$, $f = f_1 + f_2 b, h = h_1 + h_2 b$, and $h = \bar{h}$ is self-conjugated. Let R and α be as in the lemma 4. Then $f^{-1} h f = t_1 + t_2 b$, where $t_1 = h_1 + h_2(f_1 \bar{f}_2 + \bar{f}_1 \bar{f}_2) \alpha R^{-1}$, $t_2 = h_2(\bar{f}_1^2 + f_2^2 \alpha) R^{-1}$.*

Let us consider the mapping $\varphi(x_1 + x_2 b) = x_1 \bar{x}_1 + x_2 \bar{x}_2 \alpha$, where α was defined in the Lemma 5. It is easy to verify that such mapping is homomorphism from $U(KG)$ to $U(K\langle a \rangle)$ and, clearly, for every x its image $\varphi(x)$ is self-conjugated. Such mapping $\varphi : U(KG) \rightarrow U(K\langle a \rangle)$ we will call *norm*. We will also say that the norm of an element x is equal to $\varphi(x)$.

3. DIHEDRAL AND SEMIDIHEDRAL GROUP

Now let G be the dihedral or semidihedral group. Note that in Lemma 5 the first component of $f^{-1} h f$ is always self-conjugated. In general, the second one need not have the same property, but it is self-conjugated in the case when $f_1 + f_2 = 1$, where $f = f_1 + f_2 b$. It is easy to check that the set

$$H(KG) = \{h_1 + h_2 b \in U(KG) \mid h_1 + h_2 = 1\}$$

is a subgroup of $U(KG)$.

Now we define the mapping $\psi : H(KG) \rightarrow U(K\langle a \rangle)$ as a restriction of $\varphi(x)$ on $H(KG)$. For convenience we also call it *norm*.

Lemma 6. $\ker \psi = C_H(b) = \{h \in H(KG) \mid hb = bh\}$.

The proof follows from the Lemma 2 and the equality $\psi(h) = 1 + h_1 + \bar{h}_1$ for $h \in H(KG)$.

Lemma 7. $C_H(b)$ is elementary abelian group.

Proof. $C_H(b)$ is abelian by the lemma 3. Now, $h^2 = f_1^2 + f_2^2 + (f_1f_2 + f_1f_2)b = f_1^2 + f_2^2 = 1 + f_1^2 + f_1^2 = 1$, and we are done.

Note that for elements $f, h \in C_H(b)$ we may obtain more simple rule of their multiplication: $(f_1 + f_2b)(h_1 + h_2b) = (1 + f_1 + h_1) + (f_1 + h_1)b$.

Now we consider a subgroup in G generated by $b \in G$ and $A = a + (1 + a)b$. Note that $A \in H(KG)$. First we calculate the norm of A : $\psi(A) = 1 + a + \bar{a}$. Since $a^{2^{n-1}} = 1$, we have $\psi(A)^{2^{n-2}} = 1$. Then $A^{2^{n-2}}$ commute with b , and $A^{2^{n-1}} = 1$ by the Lemma 7. Smaller powers of A have non-trivial norm, so they do not commute with b . Clearly, order of A is equal to 2^{n-2} or 2^{n-1} . In the following lemma we will show that actually only the second case is possible.

Lemma 8. Let $A = a + (1 + a)b$. Then the order of A is equal to 2^{n-1} .

Proof. We will show that the case of 2^{n-2} is impossible since $\text{Supp } A^{2^{n-2}} \neq 1$. We will use formula (8) from [6], which describes 2^k -th powers of an element $x \in U(G)$:

$$x^{2^k} = x_1^{2^k} + (x_2\bar{x}_2)^{2^{k-1}}b^{2^k} + \sum_{i=1}^{k-1} (x_2\bar{x}_2)^{2^{i-1}}(x_1 + \bar{x}_1)^{2^k-2^i}b^{2^i} + x_2(x_1 + \bar{x}_1)^{2^k-1}b.$$

Let us show that the second component of $A^{2^{n-2}}$ is non-trivial. Let us denote it by $t_2(A^{2^{n-2}}) = t_2$. By the cited above formula, $t_2 = (1 + a)(a + \bar{a})^{2^{n-2}-1}$, where $\bar{a} = a^{-1}$ for the dihedral group, and $\bar{a} = a^{-1+2^{n-2}}$ for the semidihedral group.

Now we consider the case of the dihedral group. We have

$$\begin{aligned} t_2 &= (1 + a)(a + a^{-1})^{2^{n-2}-1} = (1 + a)(a^{-1}(1 + a^2))^{2^{n-2}-1} \\ &= (1 + a)a^{2^{n-2}+1}(1 + a^2)^{2^{n-2}-1} = (a^{2^{n-2}+1} + a^{2^{n-2}+2})(1 + a^2)^{2^{n-2}-1}. \end{aligned}$$

Note that if $X = \langle x \rangle$, $x^{2^m} = 1$, then $(1 + x)^{2^m-1} = \sum_{y \in X} y = \overline{X}$, where for a set X we denote by $\overline{X}$ the sum of all its elements [2]. Thus, we have

$$(1 + a^2)^{2^{n-2}-1} = 1 + a^2 + a^4 + \dots + a^{2^{n-1}-2} = \overline{\langle a^2 \rangle} = \overline{G'}.$$

Then

$$t_2 = (a^{2^{n-2}+1} + a^{2^{n-2}+2})\overline{\langle a^2 \rangle} = a\overline{\langle a^2 \rangle} + \overline{\langle a^2 \rangle} = \overline{\langle a \rangle},$$

and for the case of the dihedral group the lemma is proved.

Now we will consider the semidihedral group. We have

$$\begin{aligned} t_2 &= (1 + a)(a + a^{2^{n-2}-1})^{2^{n-2}-1} = (1 + a)(a(1 + a^{2^{n-2}-2}))^{2^{n-2}-1} \\ &= (a^{2^{n-2}-1} + a^{2^{n-2}})(1 + a^{2^{n-2}-2})^{2^{n-2}-1}, \end{aligned}$$

and the rest part of the proof is similar. Note that from $t_2(A^{2^{n-2}}) = \overline{\langle a \rangle}$ we can immediately conclude that its first component is $1 + \overline{\langle a \rangle}$, since $A \in H(G)$.

To construct a section isomorphic to the desired wreath product, first we take elements $b, b^A, b^{A^2}, \dots, b^{A^{2^{n-2}-1}}$. For every k we have $(b^{A^k})^2 = 1$. By the Lemma 5 all elements b^{A^k} are self-conjugated, since $A \in H(KG)$, and they commute each with other by the Lemma 3. So, we get the next lemma.

Lemma 9. $\langle b, b^A, b^{A^2}, \dots, b^{A^{2^{n-2}-1}} \rangle$ is elementary abelian subgroup.

Now we can obtain elements b^{A^k} , using the Lemma 5.

Lemma 10. *Let $A = a + (1 + a)b$, $R = \psi(A) = 1 + a + \bar{a}$. Then*

$$b^{A^k} = 1 + R^k + R^k b, \quad k = 1, 2, \dots, 2^{n-2}.$$

Proof. First we obtain b^A by Lemma 5 with $h_1 = 0, h_2 = 1, f_1 = a, f_2 = 1 + a$. We get

$$\begin{aligned} b^A &= (a(1 + a) + \bar{a}(1 + \bar{a}))R^{-1} + (\bar{a}^2 + (1 + a)^2)R^{-1}b \\ &= (a + \bar{a} + a^2 + \bar{a}^2)R^{-1} + (1 + a^2 + \bar{a}^2)R^{-1}b \\ &= (R + R^2)R^{-1} + R^2R^{-1}b = 1 + R + Rb. \end{aligned}$$

Now let $b^{A^k} = 1 + R^k + R^k b$. Using the same method for $h_1 = 1 + R^k, h_2 = R^k$, we get $b^{A^{k+1}} = 1 + R^{k+1} + R^{k+1}b$, as required.

Lemma 11. *There exists following direct decomposition:*

$$\langle b, b^A, b^{A^2}, \dots, b^{A^{2^{n-2}-1}} \rangle = \langle b \rangle \times \langle b^A \rangle \times \langle b^{A^2} \rangle \times \dots \times \langle b^{A^{2^{n-2}-1}} \rangle$$

Proof. We need to verify that the product of the form $b^{i_0}(b^A)^{i_1} \dots (b^{A^k})^{i_k}$, where $k = 2^{n-2} - 1, i_m \in \{0, 1\}$ and not all i_m are equal to zero, is not equal to $1 \in G$. Clearly, multiplication by b only permute components. So, we may consider only products without b and proof that they are not equal to 1 or b .

Note that b^{A^k} are self-conjugated and lies in $H(KG)$. From this follows the rule of their multiplication:

$$(1 + R^k + R^k b)(1 + R^m + R^m b) = 1 + R^k + R^m + (R^k + R^m)b.$$

The product of more than two elements is calculated by the same way:

$$(b^A)^{i_1}(b^{A^2})^{i_2} \dots (b^{A^k})^{i_k} = 1 + i_1 R + i_2 R^2 + \dots + i_k R^k + (i_1 R + i_2 R^2 + \dots + i_k R^k)b.$$

Put $\gamma = i_1 R + i_2 R^2 + \dots + i_k R^k$ and $R = 1 + r$, where $r = a + \bar{a}$. Then γ could be written in the form $\gamma = \mu + r^{j_1} + \dots + r^{j_k}$, where $\mu \in \{0, 1\}$ and $j_1 < j_2 < \dots < j_k = i_k$. Since $(a + \bar{a})^{2^{n-2}} = 0$, r is nilpotent and its smaller powers are linearly independent, so $r^{j_1} + \dots + r^{j_k} \neq 0$. From the other side, it is easy to see that the support of r^{j_s} does not contain 1, so $r^{j_1} + \dots + r^{j_k} \neq 1$. Hence $\gamma \notin \{0, 1\}$, and the support of the product $(b^A)^{i_1}(b^{A^2})^{i_2} \dots (b^{A^k})^{i_k}$ contains elements different from 1 and b , which proves the lemma.

Now we are ready to finish the proof of Theorem 1 for the dihedral and semi-dihedral groups. It was shown that $U(KG)$ contains the semi-direct product F of $\langle b \rangle \times \langle b^A \rangle \times \langle b^{A^2} \rangle \times \dots \times \langle b^{A^{2^{n-2}-1}} \rangle$ and $\langle A \rangle$. As was proved above, the order of A is 2^{n-1} and its 2^{n-2} -th power commutes with b . From this follows that the factorgroup $F/\langle A^{2^{n-2}} \rangle$ is isomorphic to $C_2 \wr G'$, as required.

4. GENERALIZED QUATERNION GROUP

Now let G be the generalized quaternion group. First we need to calculate $\text{cl}U(G)$. In fact, we need to know only $t_L(G)$, since $\text{cl}U(G) = t_L(G) - 1$ [10]. Note that Theorem 2 is already known (see Theorem 4.3 in [5]), but we provide an independent proof for the generalized quaternion group.

Theorem 2. *Let G be the generalized quaternion group. Then $\text{cl}U(G) = |G'|$.*

Proof. First, $\text{cl}U(G) \leq |G'|$ by [26]. Now we prove that $t_L(G) \geq |G'| + 1$. To do this, we will construct non-trivial Lie-product of the length $2^{n-2} = |G'|$.

Consider Lie-product $[b, \underbrace{a, \dots, a}_k]$ which we denote by $[b, k \cdot A]$. Clearly, $[b, a] = (a + a^{-1})b$, and $(a + a^{-1})$ is central in KG . It is easy to prove by induction that $[b, k \cdot a] = (a + a^{-1})^k b$, therefore the commutator

$$[b, (2^{n-2} - 1) \cdot a] = a^{2^{n-2}+1}(1 + a^2)^{2^{n-2}-1}b = a^{2^{n-2}+1}\overline{\langle a^2 \rangle}b = a\overline{\langle a^2 \rangle}b$$

does not vanish.

From the Theorem 2 it follows that $t_L(G) = t^L(G)$ since

$$\text{cl } U(G) = t_L(G) - 1 \leq t^L(G) - 1 \leq |G'|,$$

confirming conjecture about equality of the lower and upper Lie nilpotency indices (cf. [3]). From this we conclude that G and $U(KG)$ have the same exponent, using the theorem from [24] about coincidence of their exponents in the case when $t^L(G) \leq 1 + (p-1)p^{e-1}$, where $p^e = \exp G$ and p is the characteristic of the field K . Note that these two statements regarding Lie nilpotency indices and exponent are also true for all 2-groups of maximal class. Using the technique described here we also may show that modular group algebras of 2-groups of maximal class are Lie centrally metabelian.

For a unit A of KG we denote by $(b, k \cdot A)$ the commutator $(b, \underbrace{A, \dots, A}_k)$. Now

we need a pair of technical lemmas.

Lemma 12. *Let $A \in U(KG)$, $A^{2^{n-1}} = 1$, $b \in G$, $b^{A^i}b^{A^j} = b^{A^j}b^{A^i}$ for every i, j , where $b^{A^i} = A^{-i}bA^i$. Then for every $k \in \mathbf{N}$ $(b, k \cdot A)^2 = 1$.*

Proof. We use induction by k . By straightforward calculation, $(b, A)^2 = 1$. Now, let $(b, k \cdot A) = X$, $X^2 = 1$. Then $(X, A) = XX^A$. Since elements b^{A^i} , $i \in \mathbf{N}$ commute each with other, X and X^A also commute, and $(XX^A)^2 = 1$.

Lemma 13. *Let $A \in U(KG)$, $A^{2^{n-1}} = 1$, $b \in G$, $b^{A^i}b^{A^j} = b^{A^j}b^{A^i}$ for every i, j , where $b^{A^i} = A^{-i}bA^i$. Then for every $k, m \in \mathbf{N}$*

$$(b, \underbrace{A, \dots, A}_k, A^{2^m}) = (b, \underbrace{A, \dots, A}_{k+2^m}).$$

Proof. We use induction by m . First, $(b, k \cdot A, A) = (b, (k+1) \cdot A)$. Let the statement holds for some m . Consider the commutator

$$(b, k \cdot A, A^{2^{m+1}}) = (b, k \cdot A, A^{2^m})^2(b, k \cdot A, A^{2^m}),$$

since $(x, yz) = (x, y)(x, z)(x, y, z)$. By the Lemma 12 the square of the first commutator is 1, while the second is equal to $(b, (k + 2^{m+1}) \cdot A)$.

This gives possibility to proof the next property of $U(KG)$.

Lemma 14. *Let $A \in U(KG)$, $A^{2^{n-1}} = 1$, $b \in G$, $b^{A^i}b^{A^j} = b^{A^j}b^{A^i}$ for every i, j , where $b^{A^i} = A^{-i}bA^i$. Then $A^{2^{n-2}}$ commute with b .*

Proof. We will show using induction by m that the group commutator $(b, A^{2^m}) = (b, \underbrace{A, \dots, A}_{2^m})$, so $(b, A^{2^{n-2}}) = (b, \underbrace{A, \dots, A}_{2^{n-2}}) = 1$, since $\text{cl } U(G) = 2^{n-2}$.

First, $(b, A^2) = (b, A)^2(b, A, A) = (b, A, A)$ by the Lemma 12. Let $(b, A^{2^m}) = (b, 2^m \cdot A)$. Then $(b, A^{2^{m+1}}) = (b, A^{2^m})^2(b, A^{2^m}, A^{2^m}) = (b, A^{2^m}, A^{2^m})$ by the Lemma 12. Using induction hypothesis and Lemma 13, we get

$$(b, A^{2^m}, A^{2^m}) = (b, 2^m \cdot A, A^{2^m}) = (b, 2^{m+1} \cdot A).$$

Let us take an element $A = a^{2^{n-3}+1} + (1+a)b$, where $a^{2^{n-1}} = 1$. Calculating $A^{-1}hA$ for self-conjugated h by the Lemma 5, we get a self-conjugated element again. The norm of A is $\varphi(A) = 1 + a^{2^{n-2}+1} + a^{2^{n-2}-1}$, so order of $\varphi(A)$ is 2^{n-2} , and from this we conclude that the order of A is great or equal to 2^{n-2} . From the other side, it is not greater then 2^{n-1} , since G and $U(KG)$ have the same exponent. Moreover, if $A^{2^{n-2}} \neq 1$, then $A^{2^{n-2}}$ commute with b by lemma 14, and it is necessary to know whether its lower powers commute with b . As in the previous section, in the following lemma we will exactly calculate the order of A .

Lemma 15. *Let $A = a^{2^{n-3}+1} + (1+a)b$. Then the order of A is equal to 2^{n-1} .*

Proof. The proof is similar to the proof of the lemma 8. We will show that $\text{Supp } A^{2^{n-2}} \neq 1$, calculating the second component $t_2(A^{2^{n-2}}) = t_2$. Using the same formula from [6], we have:

$$\begin{aligned} t_2 &= (1+a)(a^{2^{n-3}+1} + a^{-2^{n-3}-1})^{2^{n-2}-1} = (1+a)(a^{-2^{n-3}-1}(1+a^{2^{n-2}+2}))^{2^{n-2}-1} \\ &= (1+a)(a^{-2^{n-3}-1})^{2^{n-2}-1}(1+a^{2^{n-2}+2})^{2^{n-2}-1} \\ &= (1+a)a^{-2^{n-3}+1}(1+a^{2^{n-2}+2})^{2^{n-2}-1} \\ &= (a^{-2^{n-3}+1} + a^{-2^{n-3}+2})(1+a^{2^{n-2}+2})^{2^{n-2}-1}. \end{aligned}$$

Then, $(1+a^{2^{n-2}+2})^{2^{n-2}-1} = \overline{a^2} = \overline{G'}$. From this

$$t_2 = (a^{-2^{n-3}+1} + a^{-2^{n-3}+2})\overline{a^2} = a\overline{a^2} + \overline{a^2} = \overline{a},$$

and the lemma is proved.

Now we calculate elements b^{A^k} , $k = 1, 2, \dots, 2^{n-2}$, using Lemma 5.

Lemma 16. *Let $A = a^{2^{n-3}+1} + (1+a)b$, $R = \varphi(A) = 1 + a^{2^{n-2}+1} + a^{2^{n-2}-1}$. Then*

$$b^{A^k} = \beta \sum_{i=-1}^{k-1} (b^2 R)^i + (b^2 R)^k b, \quad k = 1, 2, \dots, 2^{n-2},$$

where $\beta = a^{2^{n-3}+1} + a^{-2^{n-3}-1} + a^{2^{n-3}+2} + a^{-2^{n-3}-2}$.

Proof. Remember that for Q_n in Lemma 5 $f^{-1}hf = t_1 + t_2b$, where

$$t_1 = h_1 + h_2(f_1f_2 + \bar{f}_1\bar{f}_2)b^2R^{-1}, \quad t_2 = h_2(\bar{f}_1^2 + f_2^2b^2)R^{-1}.$$

First we obtain the second component. For $A = f_1 + f_2b$ we have $\bar{f}_1^2 + f_2^2b^2 = (a^{-2^{n-3}-1})^2 + (1+a)^2a^{2^{n-2}} = b^2(1+a^2+a^{-2}) = b^2R^2$. Then the second component of b^A is $b^2R^2R^{-1} = b^2R$. Now it is easy to prove by induction that the second component of b^{A^k} is $(b^2)^kR^k$. From this immediately follows that $A^k, k < 2^{n-2}$, doesn't commute with b , since $\text{ord } R = 2^{n-2}$.

Now we will calculate the first component. First, for the element A expression of the form $f_1f_2 + \bar{f}_1\bar{f}_2$ is equal to $a^{2^{n-3}+1}(1+a) + a^{-2^{n-3}-1}(1+a^{-1}) = a^{2^{n-3}+1} + a^{-2^{n-3}-1} + a^{2^{n-3}+2} + a^{-2^{n-3}-2}$, which we will denote by β . Using the formula at the beginning of the proof for $h_1 = 0, h_2 = 1$ we conclude that the first component of b^A is equal to βb^2R^{-1} . Now let the first component of b^{A^k} , where $k < 2^{n-2} - 1$, is equal to $\beta \sum_{i=-1}^{k-1} (b^2 R)^i$. Taking into consideration its previously calculated second

component, we obtain that the first component of $b^{A^{k+1}}$ is equal to

$$\beta \sum_{i=-1}^{k-1} (b^2 R)^i + \beta (b^2 R)^k b^2 R^{-1} = \beta \sum_{i=-1}^k (b^2 R)^i.$$

Now we are ready to construct the subgroup, whose factorgroup is isomorphic to the desired wreath product. Let us consider the subgroup F_1 in $U(G)$:

$$F_1 = \langle b, b^A, b^{A^2}, \dots, b^{A^{2^{n-2}-1}} \rangle \langle A \rangle,$$

where $b \in G$, $A = a^{2^{n-3}+1} + (1+a)b$, $A^{2^{n-2}}$ is the minimal power of A which commutes with b . Further, the subgroup $\langle b, b^A, b^{A^2}, \dots, b^{A^{2^{n-2}-1}} \rangle$ is abelian, and the intersection of subgroups $\langle b \rangle, \langle b^A \rangle, \dots, \langle b^{A^{2^{n-2}-1}} \rangle$ is $\langle b^2 \rangle$. Moreover, the order of A is 2^{n-1} .

Let us take F_2 as a factorgroup of the group F_1 as follows:

$$F_2 = F_1 / \langle b^2 \rangle \langle A^{2^{n-2}} \rangle.$$

It is clear, that $\text{cl } F_2 \leq 2^{n-2} = \text{cl } U(G)$. If we will show that actually we have equality $\text{cl } F_2 = 2^{n-2} = \text{cl}(C_2 \wr G')$, then from this it will follow that $F_2 \cong C_2 \wr G'$.

Let $M = C_2 \wr G'$ and $\text{cl } F_2 = 2^{n-2} = \text{cl } M$. Let us assume that $F_2 \not\cong M$. Then there exists such normal subgroup $N \triangleleft M$, that $M/N \cong F_2$, since there exists a homomorphism $M \rightarrow F_2$, which is induced by mapping of generators of M into F_2 . Since $|Z(M)| = 2$, $N \triangleleft M$, then $N \cap Z(M) \neq \emptyset$, so $Z(M) \subseteq N$. In this case the nilpotency class $\text{cl } F_2$ should be less then $\text{cl } M$, and we will get a contradiction.

To obtain the lower bound for the nilpotency class $\text{cl } F_2$ we will show that the commutator $(b, \underbrace{A \dots A}_{2^{n-2}-1})$ in $U(G)$ does not belong to the subgroup $\langle b^2 \rangle \langle A^{2^{n-2}} \rangle$, so

$$\text{its image in } F_2 \text{ is nontrivial. By the lemma 13 } (b, \underbrace{A \dots A}_{2^{n-2}-1}) = (b, \underbrace{A \dots A}_{2^{n-2}-1}, A^{2^{n-3}}) = (b, \underbrace{A \dots A}_{2^{n-2}-1}, A^{2^{n-4}}, A^{2^{n-3}}) = \dots = (b, A, A^2, A^4, \dots, A^{2^{n-4}}, A^{2^{n-3}}), \text{ and we obtain}$$

more simple commutator of the length $n-1$. Further, $(b, A) = b^{-1}b^A = bb^A$, then $(b, A, A^2) = bb^A b^{A^2} b^{A^3}$, and, by induction,

$$(b, A, A^2, \dots, A^{2^{n-3}}) = bb^A b^{A^2} \dots b^{A^{2^{n-2}-1}} = (bA^{-1})^{2^{n-2}} A^{2^{n-2}}.$$

It remains to show that $(bA^{-1})^{2^{n-2}}$ does not contained in $\langle b^2 \rangle \langle A^{2^{n-2}} \rangle$. Note that $(bA^{-1})^{-1} = Ab^3$, $(Ab^3)^{2^{n-2}} = (Ab)^{2^{n-2}}$.

By the Lemma 15 the second component of $A^{2^{n-2}}$ is equal to $\overline{\langle a \rangle}$. Note that it is not changed under multiplication of $A^{2^{n-2}}$ by b^2 . The same method could be used for calculation of $(Ab)^{2^{n-2}}$. We have

$$Ab = (1+a)b^2 + a^{2^{n-3}+1}b = (a^{2^{n-2}} + a^{2^{n-2}+1}) + a^{2^{n-3}+1}b.$$

Then by the formula from [6] the second component of $(Ab)^{2^{n-2}}$ is equal to

$$\begin{aligned} a^{2^{n-3}+1}(a^{2^{n-2}+1} + a^{2^{n-2}-1})^{2^{n-2}-1} &= a^{2^{n-3}+1}(a^{2^{n-2}-1}(1+a^2))^{2^{n-2}-1} \\ &= a^{2^{n-3}+1}(a^{2^{n-2}-1})^{2^{n-2}-1}(1+a^2)^{2^{n-2}-1} = a^{2^{n-3}+2}\overline{\langle a^2 \rangle} = \overline{\langle a^2 \rangle}. \end{aligned}$$

Thus, support of the second component of $(Ab)^{2^{n-2}}$ does not coincide with the support of the second component of $(A)^{2^{n-2}}$ and does not changes under multiplication of $(Ab)^{2^{n-2}}$ by b^2 . From this we conclude that $(Ab)^{2^{n-2}} \notin \langle b^2 \rangle \langle A^{2^{n-2}} \rangle$. This proves that the commutator $(b, \underbrace{A \dots A}_{2^{n-2}-1})$ also does not lies there. That is why

$\text{cl } F_2 = 2^{n-2} = \text{cl}(C_2 \wr G')$, and $F_2 \cong C_2 \wr G'$, so the theorem is proved.

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