

ON PARTIALLY PSEUDO SYMMETRIC K -CONTACT RIEMANNIAN MANIFOLDS

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ABSTRACT. A Riemannian manifold (M, g) is semi-symmetric if $(R(X, Y) \circ R)(U, V, W) = 0$. It is called pseudo-symmetric if $R \circ R = \mathfrak{F}$, $\mathfrak{F}$ being a given function of $X, \dots, W$ and g . It is called partially pseudo-symmetric if this last relation is fulfilled by not all values of $X, \dots, W$. Such manifolds were investigated by several mathematicians: I.Z. Szabó, S. Tanno, K. Nomizu, R. Deszcz and others.

In this paper we investigate K -contact Riemannian manifolds. In these manifolds the structure vector field ξ plays a special role, and this motivates our interest in the partial pseudo-symmetry of these manifolds. We also investigate the case when $R \circ R$ is replaced by $R \circ S$ (S being the Ricci tensor). We obtain conditions in order that our manifold be: (1) Sasakian or Sasakian of constant curvature 1 (in case of $R \circ R$); (2) an Einstein manifold (in case of $R \circ S$). – Our investigation is closely related to the results of S. Tanno.

1. INTRODUCTION

A Riemannian manifold (M, g) is called *locally symmetric* if its curvature tensor R is parallel: $\nabla R = 0$ (∇ denotes the Levi-Civita connection). As a proper generalization of locally symmetric manifolds the notion of *semi symmetric* manifolds was defined by

$$(1) \quad (R(X, Y) \circ R)(U, V, W) = 0, \quad X, \dots, W \in \mathfrak{X}(M)$$

(or shortly by $R \circ R = 0$) and studied by many authors, e.g. [6], [5], [7], [11], [13]). A complete intrinsic classification of these spaces was given by Z.I. Szabó [9].

It is interesting to investigate the semi-symmetry of special Riemannian manifolds. K. Nomizu proved [6] that if M is a complete, connected hypersurface in a Euclidean space R^{n+1} ($n > 3$) and it satisfies $R \circ R = 0$, then it is locally symmetric. For the case of a compact Kähler manifold M. Ogawa [7] proved that if it is semi-symmetric, then it must be locally-symmetric, that is there exists no proper semi-symmetric compact Kähler manifold. In the case of contact manifolds S. Tanno [11]–[13] showed among others that there exists no proper semi symmetric (or semi-Ricci-symmetric) K -contact manifold. Namely in K -contact manifolds $R \circ R = 0 \Rightarrow \nabla R = 0$, and $R \circ S = 0 \Rightarrow \nabla S = 0$, where S denotes the Ricci curvature tensor.

R. Deszcz and others [2]–[4] weakened the notion of semi-symmetry (respectively semi-Ricci-symmetry) and introduced the notion of *pseudo-symmetric* spaces by

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requiring in place of $R \circ R = 0$ (respectively $R \circ S = 0$) the relation

$$(P) \quad (R(X, Y) \circ R)(U, V, W) = f[(X \wedge Y) \circ R](U, V, W)$$

only, where $f \in C^\infty(M)$. – S. Tanno [11], [13], M. Okumura [8], T. Takahashi [10] and others investigated cases, where (1) is satisfied only by certain special vector fields, e.g. (I) $Y = U = W = \xi$, (II) $Y = U = \xi$ or (III) $Y = \xi$. These spaces need not be semi-symmetric. We call them *partially semi symmetric* or *partially pseudo symmetric*, as the restrictions refer to (1), respectively (P).

In this paper we consider partially pseudo-symmetric K -contact manifolds, where (P) is satisfied with the restrictions (I), (II) or (III) only. Since here f need not be zero, these spaces are more general, than the corresponding partially semi-symmetric spaces. We want to show that even for these spaces – i.e. for spaces with the weaker condition of partial pseudo-symmetry – similar conclusions can be drawn as in the case of partial semi-symmetry. If

$$(N) \quad f + 1 \text{ never vanishes,}$$

then in case of (I) a partially pseudo-symmetric K -contact manifold turns out to be Sasakian (Theorem 1); in case of (II) it is moreover of constant curvature 1, and hence it is locally symmetric (Theorem 3). We also obtain that among the K -contact Riemannian manifolds exactly the partially pseudo-symmetric ones with restriction (I) and (N) are Sasakian (Theorem 2). In the last Section 4 we prove that a partially pseudo-Ricci-symmetric K -contact manifold with the restriction $Y = V = \xi$ and with never vanishing function $f - 1$ is an Einstein manifold (Theorem 4), and we show that the relation $R(X, \xi) \circ S = (X \wedge \xi) \circ S$ holds in every Sasakian manifold (Theorem 5).

2. PRELIMINARIES

Let $(M, \varphi, \xi, \eta, g)$ be an n dimensional contact Riemannian manifold. It satisfies the relations

$$(2) \quad \eta(X) = \langle X, \xi \rangle, \quad \eta(\xi) = 1, \text{ where } \langle X, \xi \rangle \doteq g(X, \xi)$$

(We list only those basic relations which will be used in the sequel.) A contact manifold is said to be K -contact, if ξ is a Killing vector field [11]. In this case we have [11], [1], [5]:

$$(3) \quad \nabla_X \xi = -\varphi X$$

$$(4) \quad R(X, \xi)\xi = -X + \eta(X)\xi$$

$$(5) \quad \langle R(X, \xi)Y, \xi \rangle = -\langle X, Y \rangle + \eta(X)\eta(Y)$$

$$(6) \quad S(X, \xi) = 2n\eta(X).$$

If a K -contact manifold satisfies the relation

$$(7) \quad R(X, \xi)Y = -\langle X, Y \rangle\xi + \eta(Y)X,$$

then M is called a Sasakian manifold ([5], p. 273, Theorem 5.2).

The endomorphism $X \wedge Y$ and the homeomorphisms $R \circ R$ and $(X \wedge Y) \circ R$ are defined by

$$(8) \quad (X \wedge Y)Z \doteq \langle Y, Z \rangle X - \langle X, Z \rangle Y$$

$$(9) \quad (R(X, Y) \circ R)(U, V, W) \doteq R(X, Y)(R(U, V)W - R(R(X, Y)U, V)W - R(U, R(X, Y)V)W - R(U, V)(R(X, Y)W)$$

$$(10) \quad ((X \wedge Y) \circ R)(U, V, W) \doteq (X \wedge Y)(R(U, V)W) - R((X, Y)U, V)W - R(U, (X \wedge Y)V)W - R(U, V)((X \wedge Y)W).$$

(9) and (10) are kind of derivations.

3. PARTIALLY PSEUDO-SYMMETRIC K -CONTACT MANIFOLDS

We want to investigate partially pseudo-symmetric K -contact manifolds $(M, \varphi, \xi, \eta, g)$ which satisfy (P) with certain restrictions.

A) Let at first the restriction be (I) $Y = U = W = \xi$. Then we get

$$(11) \quad R(X, \xi) \circ R(\xi, V, \xi) = f[(X \wedge \xi) \circ R(\xi, V, \xi)].$$

Such K -contact manifolds with vanishing f were considered by S. Tanno [11]. He proved that such a partially semi-symmetric space is Sasakian. – Our condition is weaker than that of Tanno, for with us f in (11) need not be zero (our space is partially pseudo-symmetric, and not partially semi-symmetric). We want to show that also from this weaker condition we can draw similar conclusions.

Theorem 1. *A partially pseudo-symmetric K -contact manifold, where (P) is restricted by (I) $Y = U = W = \xi$ and $f + 1$ never vanishes (i.e. condition (N) is satisfied) is a Sasakian manifold.*

Proof. We denote the expression in the bracket on the right-hand side of (11) by A , and we calculate it. By (10)

$$A = (X \wedge \xi)(R(\xi, V)\xi) - R((X \wedge \xi)\xi, V)\xi - R(\xi, (X \wedge \xi)V)\xi - R(\xi, V)((X \wedge \xi)\xi).$$

Using for the individual terms of A the relations of Section 2 we obtain

$$\begin{aligned} (X \wedge \xi)(R(\xi, V)\xi) &\stackrel{(4)}{=} (X \wedge \xi)(V - \eta(V)\xi) \stackrel{(8)}{=} \langle \xi, V \rangle X - \langle X, V \rangle \xi \\ &\quad - \eta(V)(\langle \xi, \xi \rangle X - \langle X, \xi \rangle \xi) \stackrel{(2)}{=} -\langle X, V \rangle \xi + \eta(V)\eta(X)\xi, \\ R((X \wedge \xi)\xi, V)\xi &\stackrel{(2),(8)}{=} R(X - \eta(X)\xi, V)\xi \stackrel{(4)}{=} R(X, V)\xi - \eta(X)(V - \eta(V)\xi), \\ R(\xi, (X \wedge \xi)V)\xi &\stackrel{(4)}{=} (X \wedge \xi)V - \eta((X \wedge \xi)V)\xi \stackrel{(8)}{=} \eta(V)X - \langle X, V \rangle \xi \\ &\quad - \eta(\eta(V)X - \langle X, V \rangle \xi)\xi \stackrel{(2)}{=} \eta(V)X - \eta(X)\eta(V)\xi, \\ R(\xi, V)((X \wedge \xi)\xi) &\stackrel{(8)}{=} R(\xi, V)(X - \eta(X)\xi) \\ &\stackrel{(4)}{=} R(\xi, V)X - \eta(X)(V - \eta(V)\xi). \end{aligned}$$

From these

$$A = -(R(X, V)\xi + R(\xi, V)X + \eta(V)X - 2\eta(X)V + \langle X, V \rangle \xi).$$

However, after similar considerations, the left-hand side of (11) leads to the same expression with opposite sign (see also Tanno [11] (2.8)). Thus (11) reduces to

$$(12) \quad -A = fA.$$

With respect to our condition (N) this can hold only if $A = 0$. However $A = 0$ yields

$$(13) \quad R(\xi, V)X = \eta(X)V - \langle X, V \rangle \xi,$$

and this means (see Tanno [11] p. 451 (2.11) that M is a Sasakian manifold. $\square$

In consequence of (12) every K -contact Riemannian manifold with restriction (I) and with $f = -1$ is partially pseudo-symmetric. So we have proved the

Proposition 1. *In every K -contact Riemannian manifold*

$$R(X, \xi) \circ R = (\xi \wedge X) \circ R \quad \text{over } (\xi, V, \xi).$$

Now we easily obtain

Theorem 2. *Among the K -contact Riemannian manifolds exactly the partially pseudo-symmetric ones with restriction (I) $Y = U = W = \xi$ are Sasakian if in (P) $f + 1$ never vanishes.*

Proof. If a K -contact Riemannian manifold M is partially pseudo-symmetric with restriction (I) and (N), then according to our Theorem 1 M is Sasakian.

Conversely, let M be a Sasakian manifold. Tanno deduces (13) from $A = 0$ by making use of (5) ([11], p. 451). Now (13) holds, since M is Sasakian. Also (5) is true, since any Sasakian manifold is K -contact. Then one can invert this consideration, and from (13) obtain $A = 0$. Thus (12) is true also with (N), and this, by repeated application of (2), (4) and (8), leads to (11), i.e. a Sasakian manifold is partially pseudo-symmetric with restriction (I) and (N). $\square$

B) Let now the restriction on (P) be (II) $Y = U = \xi$. Then we get

$$(14) \quad (R(X, \xi) \circ R)(\xi, V, W) = f[(X \wedge \xi) \circ R](\xi, V, W).$$

Such K -contact manifolds with vanishing f were considered by S. Tanno. He proved ([11], Theorem 2.3 or [13]) that for a K -contact manifold M the following four conditions are equivalent (i) M is of constant curvature 1; (ii) $\nabla R = 0$; (iii) $R(X, Y) \circ R = 0 \ \forall X, Y$; (iv) $R(X, \xi) \circ R = 0$. His proof runs as follows (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is clear. Then he assumes (iv): $(R(X, \xi) \circ R)(U, V, W) = 0$ and proves (i). At this step, however he uses $(R(X, \xi) \circ R)(\xi, V, W) = 0$ only, which is (14) with $f = 0$. We want to prove that also from our somewhat weaker condition (14), where f is not necessarily zero, one can draw a similar conclusion.

Theorem 3. *A partially pseudo-symmetric K -contact manifold, where (P) is restricted by (II): $Y = U = \xi$, and $f + 1$ never vanishes, is a locally symmetric Sasakian manifold with constant curvature 1.*

Proof. Of course the broader is the restriction on the vector fields in (P), the wider is the family of spaces satisfying the restricted (P). Thus the family of the K -contact manifolds satisfying (14) is a part of those satisfying (12). Thus (14) $\Rightarrow$ (11).

Suppose (N). Then, according to our Theorem 1, from (11) follows (13), which means that M is Sasakian.

Let us denote the expression in the bracket on the right-hand side of (14) by C and calculate it. By (10)

$$C = (X \wedge \xi)(R(\xi, V)W) - R((X \wedge \xi)\xi, V)W - R(\xi, (X \wedge \xi)V)W - R(\xi, V)((X \wedge \xi)W).$$

Using for the individual terms of C the relations of Section 2 and (13) we obtain

$$\begin{aligned} & (X \wedge \xi)(R(\xi, V)W) \stackrel{(13)}{=} (X \wedge \xi)(\eta(W)V - \langle V, W \rangle \xi) \stackrel{(8)}{=} \\ & = \eta(W)\eta(V)X - \eta(W)\langle X, V \rangle \xi - \langle V, W \rangle X + \langle V, W \rangle \eta(X)\xi, \\ & R((X \wedge \xi)\xi, V)W \stackrel{(8)}{=} R(X, V)W - \eta(X)R(\xi, V)W \stackrel{(13)}{=} \\ & = R(X, V)W - \eta(X)[\eta(W)V - \langle V, W \rangle \xi], \\ & R(\xi, (X \wedge \xi)V)W \stackrel{(13)}{=} \eta(W)((X \wedge \xi)V - \langle (X \wedge \xi)V, W \rangle \xi) \stackrel{(8)}{=} \\ & = \eta(W)\eta(V)X - \eta(W)\langle X, V \rangle \xi - \langle \eta(V)X - \langle X, V \rangle \xi, W \rangle \xi, \\ & R(\xi, V)((X \wedge \xi)W) \stackrel{(13)}{=} R(\xi, V)(\eta(W)X - \langle X, W \rangle \xi) \stackrel{(13)}{=} \\ & = \eta(W)(\eta(X)V - \langle X, V \rangle \xi) - \langle X, W \rangle (V - \eta(V)\xi). \end{aligned}$$

From these

$$C = -R(X, V)W - \langle V, W \rangle X + \langle X, W \rangle V.$$

However the left-hand side of (14) gives, after similar calculations, the same expression with an opposite sign. Thus (14) reduces to

$$-C = f(p)C, \quad \forall p \in M.$$

Since $f(p) = -1$ is now excluded, $C = 0$, i.e.

$$R(X, V)W = \langle X, W \rangle V - \langle V, W \rangle X,$$

which means that M is of constant curvature 1.

Finally the mentioned theorem of Tanno gives that $\nabla R = 0$. i.e. M is locally symmetric. $\square$

Concerning the Sasakian manifolds of constant curvature 1, with the help of the previous results and considerations we easily obtain the following

Proposition 2. *In every Sasakian manifold of constant curvature 1*

$$(15) \quad ((\xi \wedge X) \circ R)(\xi, V, \xi) \equiv 0$$

Proof. Every Sasakian manifold M is K -contact. Hence, by our Proposition 1

$$(R(X, \xi) \circ R)(\xi, V, \xi) = ((\xi \wedge X) \circ R)(\xi, V, \xi).$$

However if M is of constant curvature 1, then $R(X, \xi) \circ R = 0$ according to (iv) of Tanno. So we obtain (15). $\square$

4. PARTIALLY PSEUDO-RICCI-SYMMETRIC K -CONTACT MANIFOLDS

A K -contact manifold $(M, \varphi, \eta, \xi, g)$ is said to be *pseudo-Ricci-symmetric* if it satisfies

$$(PR) \quad \begin{aligned} (R(X, Y) \circ S)(U, V) &= f((X \wedge Y) \circ S)(U, V), \\ f &\in C^\infty(M), \quad U, V \in \mathfrak{X}(M), \end{aligned}$$

where

$$(R(X, Y) \circ S)(U, V) \doteq -S(R(X, Y)U, V) - S(U, R(X, Y)V)$$

and

$$((X \wedge Y) \circ S)(U, V) \doteq -S((X \wedge Y)U, V) - S(U, (X \wedge Y)V).$$

Thus (PR) has the following more developed form

$$(16) \quad \begin{aligned} &S(R(X, Y)U, V) + S(U, R(X, Y)V) \\ &= f[S((X \wedge Y)U, V) + S(U, (X \wedge Y)V)]. \end{aligned}$$

We want to investigate partially pseudo-Ricci-symmetric K -contact manifolds which satisfy (PR) with the restriction $Y = V = \xi$. So we have

$$S(R(X, \xi)U, \xi) + S(U, R(X, \xi)\xi) = f[S((X \wedge \xi)U, \xi) + S(U, (X \wedge \xi)\xi)].$$

Applying (4), (6) and (8) we obtain

$$(17) \quad \begin{aligned} &2n\eta(R(X, \xi)U) + S(U, X) - \eta(X)S(U, \xi) \\ &= f[\eta(U)S(X, \xi) - \langle X, U \rangle S(\xi, \xi) + S(U, X) - \eta(X)S(U, \xi)]. \end{aligned}$$

The first term of (17) becomes by (2)

$$2n\langle R(X, \xi)U, \xi \rangle \stackrel{(5)}{=} 2n(-\langle X, U \rangle + \eta(X)\eta(U)).$$

Applying again (6) in four terms of (7), after a reduction we obtain

$$S(U, X) - 2n\langle X, U \rangle = f[S(U, X) - 2n\langle X, U \rangle].$$

This can hold only if either: (a) $f = 1$ or (b) $S(U, X) = 2ng(X, U)$ (i.e. $S = 2ng$). However (b) means that M is an Einstein manifold. Thus we have proved

Theorem 4. *A partially pseudo-Ricci-symmetric K -contact manifold with the restriction $Y = V = \xi$ and with never vanishing function $f - 1$ is an Einstein manifold.*

We still want to prove the following

Theorem 5. *The relation*

$$(18) \quad (R(X, \xi) \circ S)(U, V) = ((X \wedge \xi) \circ S)(U, V)$$

is satisfied on every Sasakian manifold M .

This means that every Sasakian manifold is partially pseudo-Ricci-symmetric with the restriction $Y = \xi$ and $f = 1$.

Proof. Putting $Y = \xi$ and $f = 1$ in (16), we obtain a more developed form of (18). Since M is Sasakian we have (13). Applying this, (6) and (8), a direct computation, similar to the previous ones shows that the sides of (18) are identical. $\square$

This also shows that there exist K -contact manifolds satisfying $R(X, \xi) \circ S = (X \wedge \xi) \circ S$.

REFERENCES

- [1] D. E. Blair. *Contact manifolds in Riemannian geometry*, volume 509 of *Lecture Notes in Math.* Springer, 1976.
- [2] R. Deszcz. On pseudo-symmetric spaces. *Bull. Soc. Math. Belg. Ser. A*, 44:1–34, 1992.
- [3] R. Deszcz, F. Defever, L. Verstraelen, and L. Vrancken. On pseudo-symmetric space-times. *J. Math. Phys.*, pages 5908–5921, 1994.
- [4] R. Deszcz and W. Grycak. On some class of warped product manifolds. *Bull. Inst. Math. Acad. Sinica*, 15:311–322, 1987.
- [5] M. Kon and K. Yano. *Structures on manifolds*. World. Sci. Publ., Singapore, 1984.
- [6] K. Nomizu. On hypersurfaces satisfying a certain condition on the curvature tensor. *Tohoku Math. J.*, 20:46–69, 1968.
- [7] Y. Ogawa. A condition for a compact Kahlerian space to be locally symmetric. *Nat. Sci. Rep. Ochanomizu Univ.*, 28:21–23, 1977.
- [8] M. Okumura. Some remarks on space with certain contact structure. *Tohoku Math. J.*, 14:135–145, 1962.
- [9] Z. I. Szabó. Structure theorem on Riemannian spaces satisfying $R(X, Y) \circ R = 0$. 1. the local version. *J. Diff. Geom.*, 17:531–582, 1982.
- [10] T. Takahashi. Sasakian hypersurfaces in a space of constant curvature. *Tohoku Math. J.*, 21:644–653, 1969.
- [11] S. Tanno. Isometric immersions of Sasakian manifolds in spheres. *Kodai Math. Sem. Rep.*, 21:448–458, 1969.
- [12] S. Tanno. Hypersurfaces satisfying a certain condition on Ricci tensor. *Tohoku Math. J.*, 21:297–303, 1969.
- [13] S. Tanno. Locally symmetric K -contact Riemannian manifolds. *Proc. Japan Acad.*, 43:581–583, 1967.

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