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CONDITIONS OF ANALYTICITY FOR FUNCTIONS OF ONE COMPLEX VARIABLE

T. ROZGONYI AND M. TAR

Abstract. In this paper certain necessary and sufficient conditions are considered for the analyticity of nonlinear functions of one complex variable in terms of the their monogenity set.

Let $f\colon D \to \mathbb{C}$ be a function, continuous over the domain $D \subset \mathbb{C}$, and let $z \in D$ be its any fixed point. Put

$$\varphi_z(h) = \frac{f(z+h) - f(z)}{h},$$

defined over the domain $Q_\varepsilon = \{h \in \mathbb{C} \mid 0 < |h| < \varepsilon\}$ with $\varepsilon = \varepsilon(z)$, where $\varepsilon(z)$ denotes the distance from z to the boundary of D.

Recall that for the set of monogenity (the set of differential numbers) $\mathfrak{M}_z(f)$ of f at the point z is given by Luzin's equality [1]

$$\mathfrak{M}_z = \cap_{\varepsilon>0}\overline{\mathfrak{M}_\varepsilon(z)},$$

where $\mathfrak{M}_\varepsilon(z) = \{\xi \in \mathbb{C} \mid \xi = \varphi_z(h),\, h \in Q_\varepsilon\}$.

The following assertion gives a sufficient condition for analyticity.

Theorem 1. *Let $f\colon D \to \mathbb{C}$ be a function which is continuous on the domain D and monogenic in each everywhere dense subset E of D. If f satisfies the condition*

(a) *at any point $\xi \in \mathbb{C}$ there are at most a countable family of sets $\mathfrak{M}_z$ containing ξ,*

then f is a nonlinear analytic function over D.

Proof. Assume the contrary. Then there exists a perfect subset $P \subset D$, at the points of P f is not analytic.

The condition (a) immediately implies that the set $\mathfrak{M}_z$ is not the complete plane for all $z \in D$, with the possible exception of countable set $H \subset D$.

Let $\{\xi_k\}$ be the set of points of the plane $\mathbb{C}$ with rational coordinates, and denote $P_{n,k}$ $(n, k \in \mathbb{N})$ the subset of the points $z \in P \setminus H$ with

$$|\varphi_z(h) - \xi_k| \geq \frac{1}{n} \tag{0.1}$$

for all h satisfying $0 < |h| < \frac{1}{n}$ and $z + h \in D$.

As $\{\xi_k\}$ is an everywhere dense subset of the $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathfrak{M}_z$ is closed in $\overline{\mathbb{C}}$, it is easy to see

$$P \setminus H = \bigcup_{n,k} P_{n,k}.$$

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Moreover, since f is continuous, all the sets $P_{n,k}$ are closed.

The perfect set P is of second category (in itself), hence there exist indices $n = n_0$ and $k = k_0$ such that P_{n_0,k_0} is everywhere dense in some subset P_0 of P. Since P_{n_0,k_0} is closed, we have $P_0 = P_{n_0,k_0}$ and it can be written in the form $P_0 = P \cap G_0$, where $G_0 \subset D$ is a domain. Consider the function $g(z) = f(z) - cz$, where $c = \xi_{k_0}$. By (1) for $z \in P_0$ and $0 < |h| < \frac{1}{n_0}$ the function $g(z)$ satisfies

$$\left| \frac{g(z+h) - g(z)}{h} \right| \geq \frac{1}{n_0}. \tag{0.2}$$

If we put $z = z_1$, $z + h = z_2$ into (2), we obtain

$$|g(z_2) - g(z_1)| \geq \frac{1}{n_0} |z_2 - z_1|, \tag{0.3}$$

for all $z_2 \in K_0$ and $z_1 \in K_0 \cap P_0 := P_1$, where $K_0 \subset G_0$ is an arbitrary circle of diameter $\frac{1}{n_0}$.

Therefore, the function $g(z)$ is single leafed on the perfect set P_1 and analytical on the open set $K_0 \setminus P_1$ (if it is nonempty). By Theorem 9 [2] there exists a domain $G_1 \subset K_0$ on which the function $g(z)$ single leafed if $G_1 \cap P_1 = P_2$ is nonempty. Let us consider the inverse $z = g^{-1}(w)$ of the function $w = g(z)$ on the domain $G_1^* = g(G_1)$.

From (3) it follows that the function $g^{-1}(w)$ satisfies

$$\left| g^{-1}(w_1) - g^{-1}(w_2) \right| \leq n_0 |w_1 - w_2| \tag{0.4}$$

for any $w_1 \in P_2^* = g(P_2)$ and $w_2 \in G_1^*$. According to (4) the set $\mathfrak{M}_w(g^{-1})$ for $w \in P_2^*$ is bounded. By Theorem 2 [2] the function has a complete differential almost everywhere on P_2^*. Let us denote the corresponding subset of P_2^* by Q.

We have two cases to distinquish

Case 1. The set P_2^* is everywhere dense in a circle $K \subset G_1^*$. Since $P_2 \subset P_1$, we infer that the function $g(z)$ is single leafed in the domain $G_2 = g^{-1}(K)$.

We claim that the function $g^{-1}(w)$ is monogenic almost everywhere in K, i.e. in $Q \cap K = Q_1$.

Suppose the contrary, the function g^{-1} is not monogenic at a point $w_0 \in Q_1$. Then $\mathfrak{M}_{w_0}(g^{-1})$ is a complete circle ([2], p. 21). Put $E^* = g(E \cap G_2)$. Since the function $g(z)$ is continuous and single leafed in the domain G_2 the set E^* is everywhere dense in the circle $K = g(G_2)$.

Write

$$E_a^* = \left\{ w \in E^* \mid [g^{-1}(w)]' = a \right\},$$

where $a \in S$, S is a circle with boundary $\mathfrak{M}_{w_0}(g^{-1})$. It is easy to see that $E^* \supset \cup_a E_a^*$, therefore the sets E_a^* are disjoint. Since E^* is a countable set and S is not, there exists $a = a_0$ such that $E_{a_0}^* = \emptyset$, therefore $[g^{-1}(w)]' \neq a_0$ for any $w \in E^*$.

Let us consider the function $\psi(w) = g^{-1}(w) - a_0 w$ ($w \in K$). By our assumption $0 \notin \mathfrak{M}_w(\psi)$ for $w \in K \setminus R$, with a countable set R. It is easy to see that the function $\psi(w)$ is single leafed in an open set Δ everywhere which is dense in K. To see this it suffices to take

$$M_n = \left\{ w \in K \mid \left| \frac{\varphi(w+t) - \varphi(w)}{t} \right| \geq \frac{1}{n}, 0 < |t| < \frac{1}{n} \right\},$$

and argue as in the proof of the inequality (3).

Since for any $w \in E^*$ there exists $\psi'(w) \neq 0$, the mapping $z = \psi(w)$ preserves the orientation of each component Δ_k ($k = 1, 2, \ldots$) of the open set Δ.

Now we show that the function $z = \psi(w)$ realizes an inner mapping (in the sense of S. Stoylov) of the circle K.

Again, suppose the contrary, and let $\Delta \subset K$ ($\Delta \neq K$) be the maximal open subset of K on which the mapping $z = \psi(w)$ is inner.

Analogously, to our previous argument let us consider a subset L_0 of the perfect set $L = K \setminus \Delta$ on which the function $\psi(w)$ is single leafed. Clearly, we may assume $L_0 = K_1 \cap L$, where $K_1 \subset K$ is some circle. Since $\psi(w)$ is single leafed the set $\psi(L_0)$ is nowhere dense. Hence, by Theorem 8 [2], mapping $z = \psi(w)$ is inner in the circle K_1, which is a contradiction to the maximality of Δ.

Consequently, mapping $z = \psi(w)$ is inner in the circle K. It is well-known that an inner mapping either preserves or inverts the orientation at any point of the domain. As it is shown above, $z = \psi(w)$ preserves the orientation of the domains Δ_k ($k = 1, 2, \ldots$). On the other hand, at the point w_0 it inverts the orientation, therefore, the circle

$$\mathfrak{M}_{w_0}(\psi) = \left\{\omega \in \mathbb{C} \mid \omega = \psi_{w_0} + \psi_{\overline{w_0}} e^{-2i\beta}, \beta \in [0, 2\pi]\right\}$$

contains an inner point $\omega = 0$. Hence the Jacobian mapping

$$J(w_0) = |\psi_{w_0}|^2 - |\psi_{\overline{w_0}}|^2$$

is negative.

We have obtained a contradiction, so the function f is analytical over the domain D.

Case 2. Let P_2^* nowhere dense in the domain G_1^*. Then the function $\psi(w)$ is analytical on the open set $G_1^* \setminus P_2^*$ which is everywhere dense in the domain G_1^*, is single leafed on P_2^*. Hence, by Theorem 9 [2], function $z = \psi(w)$ realizes an inner mapping of the domain G_1^*. The remaining part of the statement can be proved analogously to Case 1.

The nonlinearity of f easily follows from the condition (a), because for a linear function $f(z) = cz + d$ we have $\mathfrak{M}_z(f) = \{c\}$ for any $z \in \mathbb{C}$. □

Remark 1. *We show that the condition (a) is also necessary for the analyticity of a nonlinear function $f\colon D \to \mathbb{C}$ defined on a domain $D \subset \mathbb{C}$.*

Indeed, for an analytic function f and for $z \in D$ we have

$$\mathfrak{M}_z(f) = \{f'(z)\}\,. \tag{0.5}$$

Assume that our assertion is not true. Then, by (5), there exist a $c \in \mathbb{C}$ and an uncountable set $M \subset D$ such that $f'(z) = c$ for $z \in M$. It is easy to see that there exists a subdomain $\overline{D_1} \subset D$ such that $M_1 = M \cap \overline{D_1}$ is an un-countable set. By the Theorem of uniqueness for analytic functions we get $f'(z) \equiv c$ ($z \in D$). It follows that $f(z) = cz + c_0$ ($z \in D$), i.e., f is a linear function, which contradicts our assumption.

We point out another property of analytic functions in the next statement.

Theorem 2. *Let $w = f(z)$ be a nonlinear function which is analytic in the domain D and let $S_0 \subset D$ be a set of points $z \in D$ with $f''(z) = 0$. Then for any closed domain $\overline{D_0} \subset D \setminus S_0$ there exists $\varepsilon > 0$ such that*

$$f'(z) \notin \mathfrak{M}_\varepsilon(z), \tag{0.6}$$

where $z_0 \in \overline{D_0}$.

Proof. First note that each subdomain $\overline{D_0} \subset D$ contains at most a finite set of points of S_0. In the opposite case by the Theorem of uniqueness for analytic functions we have $f''(z) \equiv 0$ for any $z \in D$, i.e., f is linear.

Suppose that the assertion of theorem 2 is false. Then either in some subdomain $\overline{D_0} \subset D \setminus S_0$ for any $n \in \mathbb{N}$ ($n \geq n_0$) there exists a point $z_n \in \overline{D_0}$ such that

$f'(z_n) \in \mathfrak{M}_{\frac{1}{n}}(z_n)$, or there exists $\xi_n \in \mathbb{C}$ such that

$$f'(z_n) = \frac{f(\xi_n) - f(z_n)}{\xi_n - z_n}, \tag{0.7}$$

and $0 < |\xi_n - z_n| < \frac{1}{n}$.

We shall assume that the sequence $\{z_n\}$ converges to a point $z_0 \in \overline{D_0}$. (Otherwise a convergent subsequence of $\{z_n\}$ can be considered).

Clearly, $\xi_n \to z_0$ ($n \to \infty$).

By decomposing the function f into its Taylor series in the neighbourhoods of the points z_n, (7) can be rewritten as

$$f'(z_n) = f'(z_n) + \frac{f''(z_n)}{2!}(\xi_n - z_n) + \frac{f'''(z_n)}{3!}(\xi_n - z_n)^2 + \cdots$$

From this we obtain

$$\frac{f''(z_n)}{2!} + \frac{f'''(z_n)}{3!}(\xi_n - z_n) + \cdots = 0.$$

Taking limits we conclude $f''(z_0) = 0$. But this contradicts that $f''(z) \neq 0$ for any $z \in \overline{D_0}$. □

Since for an analytic function f monogenic at the point z we have (5), the condition in Theorem 2 can be reformulated as follows:

(b) *for any closed domain* $\overline{D_0} \subset D \setminus S_0$ *there exists* $\varepsilon > 0$ *such that*

$$\mathfrak{M}_z(f) \cap \mathfrak{M}_\varepsilon(z) = \emptyset, \tag{0.8}$$

where $z_0 \in \overline{D_0}$.

Note that the condition (b) (if $\overline{D_0}$ is a circle) with certain additional restrictions is also sufficient for the analyticity of a nonlinear function.

Theorem 3. *Let* $f\colon D \to \mathbb{C}$ *be a continuous function in the domain* $D \subset \mathbb{C}$, *monogenic almost everywhere in* D, *and let* $H \subset D$ *be an countable set.*

If for any closed circle $\overline{K} \subset D$ *there exists* $\varepsilon > 0$ *such that every* $z \in \overline{K} \setminus H$ *satisfies (8), then* f *is a nonlinear function which is analytic in the domain* D.

Proof. Assume the contrary; then there exists a perfect set $P \subset D$, at the points where f is not analytic.

Let $z_0 \in P$ be an arbitrary point, $K \subset D$ the circle with centre z_0 of radius $r \leq \frac{1}{2}\rho(z, \partial D)$, K_{ε_0} the concentric circle of radius $\varepsilon_0 \leq \min\{\varepsilon, \frac{r}{2}\}$, where ε is the number in (8). (Note that (8) also holds for any ε_0 with $0 < \varepsilon_0 < \varepsilon$).

For any fixed $z \in \overline{K_{\varepsilon_0}}$ consider the function $\varphi_z(h)$, $h \in Q_{\varepsilon_0}$. We have

$$\frac{\varphi_z(h+t) - \varphi_z(h)}{t} = \frac{1}{t}\left[\frac{f(z+h+t) - f(z)}{h+t} - \frac{f(z+h) - f(z)}{h}\right] =$$

$$= \frac{1}{th(h+t)}\left\{[f(z+h+t) - f(z+h)]h - [f(z+h) - f(z)]t\right\} =$$

$$= \frac{f(z+h+t) - f(z+h)}{t(h+t)} - \frac{f(z+h) - f(z)}{h(h+t)}.$$

This implies that each differential number $\omega(\varphi_z; h)$ of the function $\varphi_z(h)$ at the point h is determined by the equality

$$\omega(\varphi_z; h) = \frac{1}{h}\omega(f; z+h) - \frac{1}{h}\varphi_z(h).$$

We show that $0 \notin \mathfrak{M}_h(\varphi_z)$ for any $h \in Q_{\varepsilon_0} \setminus H_0$, where H_0 is an countable set. Indeed, $0 \in \mathfrak{M}_h(\varphi_z)$ implies that there exists h such that

$$\varphi_z(h) \in \mathfrak{M}_{z+h}(f),$$

where $z+h \in K$, or

$$\frac{f(z+h)-f(z)}{h} \in \mathfrak{M}_{z+h}(f).$$

Putting $z+h=z'$ we have

$$\frac{f(z)-f(z')}{z-z'} \in \mathfrak{M}_{z'}(f),$$

where $z, z' \in K$. However, this contradicts the condition of theorem 2.

If the function $\varphi_z(h)$ has nonzero differential in a set everywhere dense in Q_{ε_0}, and $0 \notin \mathfrak{M}_h(\varphi_z)$ for $h \in Q_{\varepsilon_0} \setminus H_0$, analogously as in the proof of Theorem 1 we claim that $\varphi_z(h)$ (for any fixed $z \in \overline{K_{\varepsilon_0}}$) realizes an inner mapping of the domain Q_{ε_0}.

Let M be the set of points for which, in accordance with the conditions of theorem 2, there exists the differential $f'(z) = \varphi_z(0)$, and $R = \max_{|h|=\varepsilon_0} |\varphi_z(h)|$ for any $z \in \overline{D_0}$.

Since the mapping $\xi = \varphi_z(h)$ is inner in the circle $K_0 = \{h \mid |h| < \varepsilon_0\}$, we assume $|\varphi_z(h)| \leq R$ for any h, $|h| < \varepsilon_0$ and $z \in M$.

Choose an arbitrary point $z_0 \in K_{\varepsilon_0} \setminus M$. By the continuity of the function $\varphi_z(h)$ of the variable z (for any fixed h) we get

$$\lim_{z \to z_0 z \in M} \varphi_z(h) = \varphi_{z_0}(h).$$

It follows that $|\varphi_z(h)| \leq R$ for any $h \in Q_{\varepsilon_0}$ and $z \in K_{\varepsilon_0}$, i.e. the sets of monogenity $\mathfrak{M}_z(f)$ are bounded in the circle K_{ε_0}. By Lemma 11 [2] we obtain that f is analytic in the circle K_{ε_0}, which contradicts $P_0 = P \cap K_{\varepsilon_0} \neq \emptyset$.

The nonlinearity of f follows from (8), since for a linear function $f(z) = cz + c_0$ we have

$$\mathfrak{M}_z(f) = \mathfrak{M}_\varepsilon(z) = \{c\}$$

for any $z \in \mathbb{C}$. □

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T. Rozgonyi
Institute of Mathematics and Computer Sience,
College of Nyregyhza,
H4401 Nyregyhza, Pf. 166
E-mail address: rozgonyi@nyf.hu

M. Tar
Department of Mathematics,
Uzhgorod State University,
294000, Uzhgorod, Ukraina