

A SHORT REMARK ON KOLMOGOROFF NORMABILITY THEOREM

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ABSTRACT. Kolmogoroff normability theorem turns to be a characterization for the complete normability of a topological vector space by replacing the convexity hypothesis with the σ -convexity one. In particular, the well known theorem that characterizes completeness of a normed vector spaces by means of absolutely convergent series, is obtained as an easy consequence of the Theorem below.

In what follows E will denote a topological vector space (t.v.s. for short) on a field $K(=\mathbb{R} \text{ or } \mathbb{C})$. Kolmogoroff theorem about normability asserts that a t.v.s. is normable if and only if E is Hausdorff and there exists a convex bounded neighborhood U of θ_E (θ_E denotes the zero of E). By requiring that U is more than a convex bounded set, i.e. a σ -convex set, we actually obtain a characterization for the completeness; precisely we prove the following theorem.

Theorem. *Let E be a t.v.s. Then E is a Banach space if and only if E is Hausdorff and there exists a σ -convex neighborhood U of θ_E .*

So what we need is the following definition.

Definition. Let E be an Hausdorff t.v.s. and $C \subseteq E$. The set C is said to be σ -convex provided that the following condition hold:

$$\forall \{x_n\} \subseteq C, \forall \{\lambda_n\} \subseteq [0, 1] \text{ such that } \sum_{n=1}^{+\infty} \lambda_n = 1, \implies \sum_{n=1}^{+\infty} \lambda_n x_n \in C.$$

It is apparent that if C is a σ -convex set then also $C+a$ ($a \in E$) and αC ($\alpha \in K$) are σ -convex.

Proposition 1. *Let E be an Hausdorff t.v.s., and $C \subseteq E$. Then*

- (i) *if C is σ -convex then is convex and bounded;*
- (ii) *if C is σ -convex and $B \subseteq C$ is convex and closed, then B is σ -convex too;*
- (iii) *if C is σ -convex then $\text{int}(C)$ is too;*
- (iv) *let C be convex and bounded. If E (respectively C) is sequentially complete then $\overline{C}$ (respectively C) is σ -convex;*

Proof. (i) C is obviously convex. Observe now that in any t.v.s. the condition $B \subseteq E$ bounded is equivalent to requiring that for any $\{\alpha_n\} \subseteq [0, 1]$, $\alpha_n \rightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n \leq 1$, and for any $\{x_n\} \subseteq B$, then $\alpha_n x_n \rightarrow \theta_E$. So chosen $\{x_n\} \subseteq C$ and $\{\alpha_n\}$ as above, set $\alpha_0 = 1 - \sum_{n=1}^{\infty} \alpha_n$ and fix $x_0 \in C$ arbitrarily: the convergence of the series $\sum_{n=0}^{\infty} \alpha_n x_n$ concludes the argument.

(ii) Take $\{x_n\} \subseteq B$ and $\{\lambda_n\} \subseteq [0, 1]$ with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then $T_n = S_n + (1 - \sum_{i=1}^n \lambda_i)x_1 \in B \ \forall n \in \mathbb{N}$. But $S_n = \sum_{i=1}^n \lambda_i x_i \rightarrow \sum_{i=1}^{\infty} \lambda_i x_i \in C$, $(1 - \sum_{i=1}^n \lambda_i)x_1 \rightarrow \theta_E$, so T_n does converges in B to the same limit of S_n .

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(iii) Suppose that $\text{int}(C) \neq \emptyset$. Take $\{x_n\} \subseteq \text{int}(C)$, $\{\lambda_n\} \subseteq [0, 1]$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1$ and set $\sum_{n=1}^{+\infty} \lambda_n x_n = \bar{x} \in C$. If $\bar{x} \notin \text{int}(C)$ then, by a consequence of the Hahn-Banach Theorem (see Theorem 3.4, (a), of [4], p.58) there exist $T \in E^*$ and $\alpha \in \mathbb{R}$ such that $\text{Re } T(x_n) < \alpha \leq \text{Re } T(\bar{x})$ for each $n \in \mathbb{N}$. It follows that $\alpha \leq \text{Re } T(\bar{x}) = \sum_{n=1}^{+\infty} \lambda_n \text{Re } T(x_n) < \alpha$. So $\bar{x} \in \text{int}(C)$.

(iv) Let E be sequentially complete. Set $\{x_n\} \subseteq \bar{C}$, $\{\lambda_n\} \subseteq [0, 1]$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1$. If $S_n = \lambda_1 x_1 + \cdots + \lambda_n x_n = (\sum_{i=1}^n \lambda_i x_i + \sum_{i=n+1}^{+\infty} \lambda_i x_1) - \sum_{i=n+1}^{+\infty} \lambda_i x_1$ does converge, its limit must belong to $\bar{C}$ because $\sum_{i=n+1}^{+\infty} \lambda_i x_1$ goes to zero and $(\sum_{i=1}^n \lambda_i x_i + \sum_{i=n+1}^{+\infty} \lambda_i x_1) \in C$ converges too. So it is enough to verify that $\{S_n\}$ is a Cauchy sequence. Fixed a neighborhood U of θ_E , choose $V \subseteq U$ balanced neighborhood of θ_E and $\alpha > 0$ such that $\bar{C} \subseteq \alpha V$. Moreover choose k large enough such that $(\lambda_{k+1} + \lambda_{k+2} + \cdots)\alpha \leq 1$. Then for each $n > m \geq k$ we have $S_n - S_m = \lambda_{m+1} x_{m+1} + \cdots + \lambda_n x_n = (\lambda_{m+1} + \cdots + \lambda_n) \left(\frac{\lambda_{m+1}}{(\lambda_{m+1} + \cdots + \lambda_n)} x_{m+1} + \cdots + \frac{\lambda_n}{(\lambda_{m+1} + \cdots + \lambda_n)} x_n \right) \in (\lambda_{m+1} + \cdots + \lambda_n) \bar{C} \subseteq (\lambda_{m+1} + \cdots + \lambda_n) \alpha V \subseteq V$. Let now C be sequentially complete: in a similar way one can prove that $\sum_{i=1}^n \lambda_i x_i + \sum_{i=n+1}^{+\infty} \lambda_i x_1$ is a Cauchy sequence in C so that S_n does converge to some element of C . $\square$

Remark. Relatively to (i) note that not every convex bounded set is σ -convex: the set C of all complex sequences $\{x_n\}$ in the unit ball of l_∞ such that $|x_n| \leq 1$ for each $n \in \mathbb{N}$ and $x_n = 0$ for all but finitely many $n \in \mathbb{N}$, is convex and bounded, but evidently C is not σ -convex in l_∞ . Moreover, in the assertion (ii), the assumption B closed cannot be removed: the set $C = C_0^\infty(\mathbb{R}) \cap B$, being B the closed unit ball in $L^1(\mathbb{R})$, is convex bounded and dense in B , but C is not σ -convex in $L^1(\mathbb{R})$.

Proof of Theorem. By (iv) the closed unit ball is a σ -convex neighborhood of θ_E whenever E is Banach space. Let assume now that U is a σ -convex neighborhood of θ_E . By (i) of Proposition 1 and Kolmogoroff theorem we can find a norm $\|\cdot\|$ on E whose topology coincides with the given one. Let $B \subseteq U$ be a closed ball. By (ii) of Proposition 1 B is σ -convex, consequently by (iii) and translations argument, so is any other ball in E . Let $\{x_n\}$ be a Cauchy sequence and $\{x_{n_k}\}$ a subsequence such that $\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k} \forall k \in \mathbb{N}$: it is enough to prove that such a sequence converges to some element of E . Set $y_k = 2^k(x_{n_{k+1}} - x_{n_k}) \forall k \in \mathbb{N}$. It results $\{y_k\} \subseteq B$, B the open unit ball in E . By construction there exists $y \in B$ such that $y = \sum_{k=1}^{+\infty} \frac{y_k}{2^k} = \lim_k \sum_{i=1}^k (x_{n_{i+1}} - x_{n_i})$. This implies that $x_{n_k} \longrightarrow y + x_{n_1} \in E$, and the proof is complete. $\square$

It is known that if E is a finite dimensional vector space a set C is σ -convex if and only if it is convex and bounded (the if part follows by Example 1.6, iv), of [2], p.84). This is false in general as showed in the Remark. If E is a normed space, next Proposition 2 will give us a similar characterization. Recall that a set $B \subseteq E$ containing θ_E is said to be *absorbing* if for any $x \in E$ is possible to find a number $t > 0$, depending on x , such that $x \in tB$; moreover B is said to be *radial at θ_E* if for any $x \in E$ there is a number $\delta > 0$, depending on x , such that $\lambda x \in B$ for any $\lambda \in [0, \delta]$. Clearly any set B radial at θ_E is an absorbing set and it is easy to verify that any neighborhood of θ_E is radial at θ_E . Finally recall that to any absorbing set B we can associate the *Minkowski functional* p_B defined by the position $p_B(x) = \inf\{t > 0 : x \in tB\}$.

Lemma. *Let E be an Hausdorff t.v.s. and $A \subseteq E$ an open convex set. Then $A = \text{int}(\bar{A})$.*

Proof. It is enough to verify that $\text{int}(\bar{A}) \subseteq A$, i.e., $\bar{A} \setminus A \subseteq \bar{A} \setminus \text{int}(\bar{A})$. Without loss of generality we can suppose that $\theta_E \in A$. A is an open convex neighborhood

of θ_E , so the equalities $A = \{x \in E : p_A(x) < 1\}$ and $\overline{A} = \{x \in E : p_A(x) \leq 1\}$ hold (see Lemma 3.5.5, (d), of [1], p.154). Choose $c \in \overline{A} \setminus A$: it results $p_A(c) = 1$. Arguing by contradiction, suppose that $c \in \text{int}(\overline{A})$. So we can find a neighborhood U of c , $U \subseteq \overline{A}$, such that $p_A(x) \leq 1$ for any $x \in U$. Let $g :]0, +\infty[\rightarrow E$ be the function defined by the formula $g(t) = \frac{c}{t}$. It is $g(1) = c$. By the continuity of g we can find $0 < \epsilon < 1$ such that $g([1 - \epsilon, 1 + \epsilon]) \subseteq U$. It follows that $c \in (1 - \epsilon)\overline{A}$, so $p_{\overline{A}}(c) \leq 1 - \epsilon$. But $p_{\overline{A}} = p_A$ (see Theorem 1.35, (d), of [4], p.25): this contradiction concludes the argument. $\square$

Proposition 2. *Let E be a normed space. Then the following facts are equivalent:*

- (a) *E is a Banach space;*
- (b) *For any open set, convex and bounded means σ -convex;*
- (c) *The open unit ball B , or equivalently any other open ball in E , is σ -convex.*

Proof. (a) $\implies$ (b) Let $A \neq \emptyset$ be any open convex bounded set in E . By (iv) of Proposition 1, $\overline{A}$ is σ -convex and, by (iii) and the previous Lemma, so is $A = \text{int}(\overline{A})$.

(b) $\implies$ (c) It is trivial.

(c) $\implies$ (a) Following the notations of the proof of the Theorem (the only if part), if $\{x_n\}$ is Cauchy sequence, then $\{y_k\}$ belongs to the open unit ball B of E . By the hypothesis B is σ -convex: this concludes the proof. $\square$

The following two corollaries to Theorem are immediate consequences, so their easy proofs are left to the reader.

Corollary 1. *Let $(E, \|\cdot\|)$ be a normed space and B its closed unit ball centered in θ_E . Then E is complete if and only if*

$$\forall \{x_n\} \subseteq B, \forall \{\lambda_n\} \subseteq [0, 1] \text{ such that } \sum_{n=1}^{+\infty} \lambda_n = 1 \implies \sum_{n=1}^{+\infty} \lambda_n x_n \in B.$$

Corollary 2. *Let $(E, \|\cdot\|)$ be a normed space. Then E is complete if and only if every absolutely convergent series is convergent.*

Sketch. Apply Corollary 1. $\square$

We conclude the present note giving some easy application of the previous results.

Example. Let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary measure space. Consider the vector space $L^p(\mu)$, $1 \leq p < \infty$, consisting of all (classes of equivalence of) measurable functions f such that $|f|^p$ is summable over Ω with respect to the measure μ . The formula $\|f\|_p = \left(\int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}}$ defines a norm on $L^p(\mu)$ which so becomes an Hausdorff t.v.s.. In order to verify the completeness of the space we can directly apply the Theorem and verify that the closed ball B with radius one, centered at zero is actually σ -convex. Take $\{f_n\} \subset B$ and $\{\lambda_n\} \subseteq [0, 1]$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1$. By Minkowski inequality it results $\left(\int_{\Omega} \left| \sum_{n=1}^k \lambda_n f_n \right|^p d\mu \right)^{\frac{1}{p}} \leq \sum_{n=1}^k \lambda_n \leq 1 \forall k \in \mathbb{N}$, so, by the Monotone Convergence theorem, $\left(\int_{\Omega} \left| \sum_{n=1}^{\infty} \lambda_n f_n \right|^p d\mu \right)^{\frac{1}{p}} \leq 1$. Thus the function $\sum_{n=1}^{+\infty} \lambda_n f_n$, defined a.e. on Ω , belongs to B .

Analogous considerations, based on the convexity property of a Young function M , hold if we want to prove the completeness of the more general Orlicz Spaces L_M (see for instance [3]).

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