

ON A CLASS OF RICCI-RECURRENT MANIFOLDS

STANISŁAW EWERT-KRZEMIENIEWSKI

ABSTRACT. Properties of Ricci-recurrent manifolds with some conditions imposed on the Weyl conformal curvature tensor are investigated. The main theorem states: a conformally quasi-recurrent and Ricci-recurrent but not Ricci-parallel manifold of dimension $n > 4$ with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor which is non-conformally related to a conformally symmetric one must be necessary Ricci-generalized pseudosymmetric manifold.

1. INTRODUCTION

Let (M, g) be a n -dimensional semi-Riemannian manifold with metric g . A tensor field T of type $(0, q)$ is said to be recurrent ([Rot82a]) if the relation

$$(1) \quad \nabla_X T(Y_1, \dots, Y_q) T(Z_1, \dots, Z_q) - T(Y_1, \dots, Y_q) \nabla_X T(Z_1, \dots, Z_q) = 0$$

holds on (M, g) . From the definition it follows that if at a point $x \in M$, $T(x) \neq 0$, then on some neighbourhood of x there exists a unique covector field b satisfying

$$\nabla_X T(Y_1, \dots, Y_q) = b(X) T(Y_1, \dots, Y_q).$$

According to Adati and Miyazawa ([AM67]), a manifold (M, g) of dimension $n \geq 4$ is called conformally recurrent if its Weyl conformal curvature tensor C satisfies (1). It is obvious that the class of conformally recurrent manifolds contains all conformally symmetric one, i.e. all manifolds satisfying $\nabla C = 0$.

Investigating conformally flat hypersurfaces immersed in an $(n+1)$ -dimensional Euclidean space R. N. Sen and M. C. Chaki ([SC67]) found that if at least $n-1$ principal curvatures are equal to one another and the remaining one is zero, then the Riemann curvature tensor satisfies

$$\begin{aligned} \nabla_Z R(X, Y, V, W) = & 2p(Z)R(X, Y, V, W) + \\ & p(X)R(Z, Y, V, W) + p(Y)R(X, Z, V, W) + \\ & p(V)R(X, Y, Z, W) + p(W)R(X, Y, V, Z) \end{aligned}$$

for some 1-form p . Manifolds satisfying the above condition are called pseudosymmetric (pseudo-symmetric in the sense of M. Chaki) ([Cha87]). We shall call such manifold quasi-recurrent rather than pseudosymmetric since the last notion is used in a different context (see below).

A $(0, 4)$ tensor B is said to be generalized curvature one if it satisfies

$$\begin{aligned} B(X, Y, V, W) = & -B(Y, X, V, W) = B(V, W, X, Y), \\ B(X, Y, V, W) + & B(X, V, W, Y) + B(X, W, Y, V) = 0. \end{aligned}$$

2000 *Mathematics Subject Classification.* 53B20, 53B30, 53C25, 53C50.

Key words and phrases. Ricci-recurrent, Ricci-flat, conformally quasi-recurrent, conformally related, pseudosymmetry, Ricci-generalized pseudosymmetric.

If, moreover, the second Bianchi identity

$$\nabla_Z B(X, Y, V, W) + \nabla_V B(X, Y, W, Z) + \nabla_W B(X, Y, Z, V) = 0$$

holds, then B is said to be a proper generalized curvature tensor.

From the results of ([EK93], Proposition 2) it follows

Proposition 1. *Let (M, g) be a semi-Riemannian manifold and B be a generalized curvature tensor on M satisfying*

$$\nabla_{X_5} B(X_1, X_2, X_3, X_4) = \sum_{\sigma}^{\bar{\sigma}} \bar{p}(X_{\sigma(5)}) B(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}, X_{\sigma(4)}),$$

for some 1-forms $\bar{p}$, where the sum includes all permutations σ of the set $\{1, 2, 3, 4, 5\}$.

Then there exist 1-forms w, p such that

$$\begin{aligned} \nabla_Z B(X, Y, V, W) = & w(Z)B(X, Y, V, W) + \\ & p(X)B(Z, Y, V, W) + p(Y)B(X, Z, V, W) + \\ & p(V)B(X, Y, Z, W) + p(W)B(X, Y, V, Z) \end{aligned}$$

holds on M . Moreover, $w = 2p$ if and only if B is a proper generalized curvature tensor.

Thus the weakly symmetric manifolds ([TB92]) as well as the generalized pseudosymmetric one ([Cha94]) are simply quasi-recurrent.

In ([EK93]) it is also proved that conformally flat quasi-recurrent manifold is of quasi-constant curvature and is subprojective in the sense of Kagan ([Kru61]). Finally, the local form of the metric of conformally flat quasi-recurrent manifold was found ([EK93]). Similar results were obtained later by others authors (cf. for example [De00] and [CM97]).

Following Prvanović ([Prv88]), a semi-Riemannian manifold (M, g) , $\dim M \geq 4$, will be called conformally quasi-recurrent if its Weyl conformal curvature tensor C satisfies

$$(2) \quad \begin{aligned} \nabla_Z C(X, Y, V, W) = & w(Z)C(X, Y, V, W) + \\ & p(X)C(Z, Y, V, W) + p(Y)C(X, Z, V, W) + \\ & p(V)C(X, Y, Z, W) + p(W)C(X, Y, V, Z) \end{aligned}$$

for some 1-forms w, p . The forms w, p will be referred to as fundamental forms or fundamental vectors. In condition considered originally by Prvanović $w = 2p$. However, the last relation together with (2) implies that for the tensor C the second Bianchi identity must hold and, consequently, the manifold is of harmonic conformal curvature.

The aim of this paper is to investigate properties of conformally quasi-recurrent manifolds in the sense of (2) which are simultaneously Ricci-recurrent, i.e. those the Ricci tensor S satisfies

$$\nabla S = b \otimes S$$

for some 1-form b .

For a generalized curvature tensor B define endomorphism $\tilde{B}(X, Y)$ by

$$g(\tilde{B}(X, Y)V, W) = B(X, Y, V, W).$$

Then for a $(0, k)$ tensor field T , $k \geq 1$, and $(0, 2)$ tensor field S we define the tensor fields $B \cdot T$ and $Q(S, T)$ by the formulas

$$\begin{aligned} (B \cdot T)(X_1, \dots, X_k; X, Y) = & \\ & -T(\tilde{B}(X, Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, \tilde{B}(X, Y)X_k), \\ Q(S, T)(X_1, \dots, X_k; X, Y) = & \\ & -T((X \wedge_S Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_S Y)X_k), \end{aligned}$$

where

$$(X \wedge_S Y) Z = S(Y, Z)X - S(X, Z)Y.$$

If the tensors $R \cdot R$ and $Q(S, R)$ are linearly dependent then the manifold is said to be Ricci-generalized pseudosymmetric one ([DD91a]). It is obvious that any semisymmetric as well as any Ricci flat manifold is Ricci generalized pseudosymmetric. The manifold (M, g) is Ricci-generalized pseudosymmetric iff the relation

$$(3) \quad R \cdot R = LQ(S, R)$$

holds on the set $\{x \in M, Q(S, R)(x) \neq 0\}$, L being a function on M . Note that (3) with $L = 1$ is of particular importance.

All manifolds under consideration are assumed to be smooth Hausdorff connected and their metrics are not assumed to be definite.

2. PRELIMINARY RESULTS

Components of the Weyl conformal curvature tensor C are given by

$$(4) \quad C_{hijk} = R_{hijk} - \frac{1}{n-2} (g_{ij}S_{hk} - g_{ik}S_{hj} + g_{hk}S_{ij} - g_{hj}S_{ik}) + \frac{r}{(n-1)(n-2)} (g_{ij}g_{hk} - g_{ik}g_{hj})$$

In the sequel we shall often need the following lemmas:

Lemma 2. *The Weyl conformal curvature tensor satisfies the following well-known relations:*

$$C_{hijk} = -C_{ihjk} = C_{jkhi}, \quad C_{rjk}^r = C_{irk}^r = C_{rjk}^r, \\ C_{hijk} + C_{hjk i} + C_{hkij} = 0,$$

$$(5) \quad C_{ijk,r}^r = \frac{n-3}{n-2} \left[S_{ij,k} - S_{ik,j} - \frac{1}{2(n-1)} (g_{ij}r_{,k} - g_{ik}r_{,j}) \right],$$

$$(6) \quad C_{hijk,l} + C_{hikl,j} + C_{hilj,k} = \\ \frac{1}{n-3} (g_{hj}C_{ikl,r}^r + g_{hk}C_{ilj,r}^r + g_{hl}C_{ijk,r}^r - \\ g_{ij}C_{hkl,r}^r - g_{ik}C_{hlj,r}^r - g_{il}C_{hjk,r}^r),$$

where the comma denotes covariant differentiation with respect to the coordinate vector field.

Lemma 3 ([Wal50], p.26). *On every semi-Riemannian manifold the curvature tensor fulfills the relation*

$$R_{hijk,[lm]} + R_{jklm,[hi]} + R_{lmhi,[jk]} = 0.$$

The following lemma seems to be well-known:

Lemma 4 ([Ols87]). *Let M be a Ricci-recurrent manifold such that the set $U = \{x \in M, b(x) \neq 0\}$ is non-empty, b being the recurrence covector of the Ricci tensor. Then the Ricci tensor satisfies*

$$(7) \quad S_{hr}S_k^r = \frac{r}{2}S_{hk}.$$

Lemma 5 ([EK98]). *Let A_{lm} , B_{lm} , R_{hijk} be numbers satisfying*

$$A_{lm} = -A_{ml}, \quad B_{lm} = -B_{ml}, \quad R_{hijk} = -R_{ihjk} = R_{jkhi},$$

$$R_{hijk} + R_{hjk i} + R_{hkij} = 0,$$

$$A_{lm}R_{hijk} + A_{hi}R_{jklm} + A_{jk}R_{lmhi} + B_{hl}R_{mijk} + B_{il}R_{hmjk} + \\ B_{jl}R_{himk} + B_{kl}R_{hijm} - B_{hm}R_{lij k} - B_{im}R_{hljk} - B_{jm}R_{hilk} - B_{km}R_{hijl} + \\ B_{ij}R_{hk lm} - B_{ik}R_{hjlm} + B_{hk}R_{ijlm} - B_{hj}R_{iklm} = 0.$$

Then either $A_{lm} - 2B_{lm} = 0$ or $R_{hijk} = 0$.

Lemma 6 ([Rot82b]). *If w_i, p_i, R_{hijk} are numbers satisfying*

$$w_l R_{hijk} + p_h R_{lij k} + p_i R_{hljk} + p_j R_{hilk} + p_k R_{hijl} = 0,$$

$$R_{hijk} = -R_{ihjk} = R_{jkh i}, \quad R_{hijk} + R_{hjki} + R_{hki j} = 0,$$

then either each $w_l + 2p_l = 0$ or each $R_{hijk} = 0$.

Lemma 7 ([Pat81]). *On every 4-dimensional semi-Riemannian manifold (M, g) the Weyl conformal curvature tensor satisfies*

$$(8) \quad g_{hm} C_{lijk} + g_{lm} C_{ihjk} + g_{im} C_{hljk} + g_{hj} C_{likm} + g_{lj} C_{ihkm} + g_{ij} C_{hlkm} + g_{hk} C_{limj} + g_{lk} C_{ihmj} + g_{ik} C_{hlmj} = 0.$$

The next lemma shows the difference between 1-forms p and w . Let p_l and w_l be the local components of the 1-forms. Then the local form of (2) is

$$(9) \quad C_{hijk,l} = w_l C_{hijk} + p_h C_{lij k} + p_i C_{hljk} + p_j C_{hilk} + p_k C_{hijl}.$$

Lemma 8. *Suppose that at a point of the manifold M relation (9) holds. Then*

$$(10) \quad p_r C_{ijk}^r = 0,$$

$$(11) \quad w_r C_{ijk}^r = C_{ijk,r}^r.$$

Moreover

$$(12) \quad C_{hijk,[lm]} = \Delta w_{lm} C_{hijk} + p_{hm} C_{lij k} + p_{im} C_{hljk} + p_{jm} C_{hilk} + p_{km} C_{hijl} - p_{hl} C_{mijk} - p_{il} C_{hmjk} - p_{jl} C_{himk} - p_{kl} C_{hijm},$$

where $\Delta w_{lm} = w_{l,m} - w_{m,l}$, $p_{hm} = p_{h,m} - p_h p_m$.

Proof. Contracting (9) with g^{ij} and making use of Lemma 2 we obtain (10). Summing (9) cyclically in (j, k, l) , by contraction with g^{hl} and the use of (10), we get (11). Relation (12) is obvious.

Proposition 9. *Let M be a 4-dimensional manifold with nowhere vanishing Weyl conformal curvature tensor C . If C satisfies (9), then M is conformally recurrent manifold.*

Proof. We can suppose $p \neq 0$. Transvecting (8) with p^h and making use of (10) we get $p_m C_{lij k} + p_j C_{likm} + p_k C_{limj} = 0$, which reduce (9) to $\nabla C = (w + 2p) \otimes C$. This completes the proof.

In the sequel we shall often assume the following hypothesis:

(H). *M is a Ricci-recurrent manifold with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor. Moreover, the Weyl conformal curvature tensor satisfies (9), p does not vanishes on a dense subset and the Ricci tensor is not parallel.*

By hypothesis, M admits a covector field b satisfying

$$(13) \quad S_{ij,k} = b_k S_{ij}, \quad S_{ij,kl} = (b_{k,l} + b_k b_l) S_{ij}, \quad S_{ij,[kl]} = \Delta b_{kl} S_{ij},$$

where $\Delta b_{kl} = b_{k,l} - b_{l,k}$.

Remark. In what follows we can always assume $p(x) \neq 0$ whenever it is necessary, because if $p(x) = 0$, then, in virtue of (H), in each neighbourhood of x we can choose a point, say y , such that $p(y) \neq 0$ and by usual procedure extend our results to the point x .

Lemma 10. *Let M , $\dim M \geq 4$, be a Ricci-recurrent manifold with non-parallel Ricci tensor and suppose that (10) is satisfied for some non-zero covector p at a point y . Then the scalar curvature of M vanishes at y .*

Proof. The Ricci identity together with (13) and (4) gives

$$(14) \quad \Delta b_{lm} S_{hk} = S_{hr} C_{klm}^r + S_{kr} C_{hlm}^r + \frac{n-3}{2(n-1)(n-2)} r (g_{kl} S_{hm} - g_{km} S_{hl} + g_{hl} S_{km} - g_{hm} S_{kl}),$$

which, by contraction with g_{hk} , yields $\Delta b_{lm} r = 0$. Suppose $r(y) \neq 0$. By transvecting (14) with S_x^k , applying (7) and symmetrizing the resulting equation in (h, x) , we obtain

$$(15) \quad S_{hr} C_{xlm}^r + S_{xr} C_{hlm}^r + \frac{n-3}{2(n-1)(n-2)} r (g_{xl} S_{hm} - g_{xm} S_{hl} + g_{hl} S_{xm} - g_{hm} S_{xl}) = 0$$

whence, by contracting with g^{xl} and transvecting with p^h , by the use of (10), we get $p_r S_m^r = \frac{r}{n} p_m$ at y . Next, transvecting (15) with p^x , we find

$$(S_{hm} - \frac{r}{n} g_{hm}) p_l = (S_{hl} - \frac{r}{n} g_{hl}) p_m.$$

Hence, in virtue of (10), $(S_{hr} C_{pqt}^r - \frac{r}{n} C_{hpqt}) p_m = 0$ at y results. Applying the last equality to (15) we have

$$r (g_{xl} S_{hm} - g_{xm} S_{hl} + g_{hl} S_{xm} - g_{hm} S_{xl}) = 0,$$

which, by transvecting with $S^{xl} g^{hm}$, yields $r = 0$ at y , a contradiction. This completes the proof.

Lemma 10 results in

Proposition 11. *Let M , $\dim M \geq 4$, be a Ricci-recurrent manifold with non-parallel Ricci tensor and suppose that M admits a covector field p with properties:*

- i) p does not vanish on a dense subset of M ;*
- ii) $p_r C_{ijk}^r = 0$ on M .*

Then the scalar curvature of M vanishes.

As a consequence of the above Proposition under hypothesis H we have the following frequently used formulas:

$$S_{hr} S_k^r = 0, \quad S_{k,r}^r = S_k^r b_r = 0$$

Lemma 12. *Under hypothesis (H) relation*

$$(16) \quad 2S_{mr} C_{ijk}^r - S_{mi} \Delta b_{jk} = 0$$

holds on M .

Proof. By Proposition 11 and (14) we have

$$(17) \quad \Delta b_{lm} S_{hk} = S_{hr} C_{klm}^r + S_{kr} C_{hlm}^r$$

on M . Differentiating covariantly with respect to ∂_z , by the use of (13), (9) and (17), we obtain

$$(18) \quad w_z \Delta b_{lm} S_{hk} + p_r S_h^r C_{zklm} + p_r S_k^r C_{zhlm} + p_k S_{hr} C_{zlm}^r + p_h S_{kr} C_{zlm}^r + p_l \Delta b_{zm} S_{hk} + p_m \Delta b_{lz} S_{hk} = S_{hk} \Delta b_{lm,z}.$$

Summing cyclically the last equation in (h, k, z) and again making use of (17) we get

$$S_{hk} W_{lmz} + S_{kz} W_{lmh} + S_{zh} W_{lmk} = 0,$$

where $W_{lmz} = -\Delta b_{lm,z} + (w_z + p_z) \Delta b_{lm} + p_l \Delta b_{zm} - p_m \Delta b_{zl}$, which, in virtue of Lemma 3, results in

$$(19) \quad \Delta b_{lm,z} = (w_z + p_z) \Delta b_{lm} + p_l \Delta b_{zm} - p_m \Delta b_{zl}.$$

Now, the last result together with (18) yields

$$(20) \quad p_r S_h^r C_{zklm} + p_r S_k^r C_{zhlm} + p_k S_{hr} C_{zlm}^r + p_h S_{kr} C_{zlm}^r = S_{hk} p_z \Delta b_{lm},$$

whence, by transvection with S_v^k , we obtain

$$(21) \quad p_r S_h^r S_{vs} C_{zlm}^s = p_s S_v^s S_{hr} C_{zlm}^r.$$

On the other side, applying to the left hand side of (12) in turn the Ricci identity, (4) and transvecting with p^h we find

$$(22) \quad p_l S_{mr} C_{ijk}^r - p_m S_{lr} C_{ijk}^r + p_r S_l^r C_{mijk} - p_r S_m^r C_{lijk} = (n-2)(p_{rm} p^r C_{lijk} - p_{rl} p^r C_{mijk}).$$

Moreover, transvecting (22) with S_v^l , making use of (21) and symmetrizing the resulting equation in (m, i) , we get $p_{rm} p^r S_{vs} C_{ijk}^s = 0$. Suppose $p_{rm} p^r = 0$. Then (22) gives

$$(23) \quad p_l S_{mr} C_{ijk}^r - p_m S_{lr} C_{ijk}^r + p_r S_l^r C_{mijk} - p_r S_m^r C_{lijk} = 0.$$

Changing in (20) indices (h, k, z, l, m) into (l, m, i, j, k) respectively, then adding to (23) and symmetrizing in (l, i) , we get $p_l (2S_{mr} C_{ijk}^r - S_{mi} \Delta b_{jk}) = 0$. On the other side, if $S_{vr} C_{ijk}^r = 0$, then (17) implies $S_{vi} \Delta b_{jk} = 0$. Thus the Lemma is proved.

Lemma 13. *Under hypothesis (H) relations*

$$p_r S_h^r = 0,$$

$$\Delta b = 0,$$

$$S_{mr} C_{ijk}^r = 0$$

hold on M.

Proof. Transvecting (16) with p^i , by the use of (10), we have

$$(24) \quad p_r S_m^r \Delta b_{jk} = 0.$$

Differentiating covariantly (16), in view of (13), (9) and (19), we find

$$(25) \quad 2p_r S_m^r C_{lijk} + (p_i S_{ml} - p_l S_{mi}) \Delta b_{jk} = 0.$$

Hence, if $\Delta b_{jk} = 0$, then $p_r S_m^r = 0$ holds. On the other hand, if $\Delta b_{jk} \neq 0$, then (24) yields $p_r S_m^r = 0$ again. Then, by the use of (25), we have $(p_i S_{ml} - p_l S_{mi}) \Delta b_{jk} = 0$. Transvecting the last result with C_{abc}^l , in virtue of (16) and (10), we get $\Delta b_{jk} = 0$. This completes the proof.

Corollary 14. *Under hypothesis (H)*

$$d(w + 2p) = 0$$

holds on M.

Proof. This results from (12) by the use of Lemmas 3, 13 and 5.

Proposition 15. *Assume that on a manifold (M, g) hypothesis (H) is satisfied. If $t = w - 2p = 0$ on (M, g) , then the manifold is conformally related to a non-conformally flat conformally symmetric one $(M, \exp(2f)g)$. Conversely, if (M, g) is conformally related to a non-conformally flat conformally symmetric one, then $w - 2p = 0$.*

Proof. It is well known that the Christoffel symbols and the Weyl conformal curvature tensor of the conformally related manifolds $(\bar{M}, \bar{g}) = (M, \exp(2f)g)$ and (M, g) are related by

$$\overline{\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}} = \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} + \delta_j^i f_k + \delta_k^i f_j - f^i g_{jk}, \quad \bar{C}_{hijk} = e^{2f} C_{hijk}.$$

By the above formulas and (H) covariant derivatives of C and $\bar{C}$ with respect to the appropriate metrics satisfy

$$(26) \quad \begin{aligned} e^{-2f} \bar{C}_{hijk;l} = \\ (w_l - 2f_l) C_{hijk} + \\ (p_h - f_h) C_{lij k} + (p_i - f_i) C_{hljk} + (p_j - f_j) C_{hilk} + (p_k - f_k) C_{hijl} + \\ g_{hl} f^r C_{rij k} + g_{il} f^r C_{hrjk} + g_{jl} f^r C_{hirk} + g_{kl} f^r C_{hijr}, \end{aligned}$$

where $f_l = \partial_l f$, $f^l = g^{lr} f_r$.

Suppose $t = w - 2p = 0$. By Corollary 14 we have $dw = dp = 0$. Hence on a neighbourhood of each point there exists a function, say f , such that $f_l = \partial_l f = p_l$ which, in virtue of (10), gives $\bar{\nabla} \bar{C} = 0$.

On the other hand, by transvecting (26) with p^h , in view of (10) and $\bar{\nabla} \bar{C} = 0$, we obtain

$$p^r (p_r - f_r) C_{lij k} + p_l f^r C_{rij k} = 0,$$

which by symmetrization in (l, i) yields $f^r C_{rij k} = 0$. Then relation (26) and Lemma 6 imply $w_l + 2p_l = 4f_l$. But $f^r C_{rij k} = 0$, (10) and (11) result in $C_{ijk,r}^r = 0$ which, together with (6), means that C satisfies the second Bianchi identity. The last property is equivalent to $w = 2p$. This completes the proof.

Lemma 16. *Suppose that hypothesis (H) is satisfied. Then on M relations*

$$(27) \quad t_r C_{pqt}^r = \frac{n-3}{n-2} (S_{pq} b_t - S_{pt} b_q)$$

and

$$(28) \quad \begin{aligned} t_l C_{hijk} + t_j C_{hikl} + t_k C_{hilj} = \\ \frac{1}{n-2} [g_{hj} (S_{ik} b_l - S_{il} b_k) + g_{hk} (S_{il} b_j - S_{ij} b_l) + g_{hl} (S_{ij} b_k - S_{ik} b_j) - \\ g_{ij} (S_{hk} b_l - S_{hl} b_k) - g_{ik} (S_{hl} b_j - S_{hj} b_l) - g_{il} (S_{hj} b_k - S_{hk} b_j)] \end{aligned}$$

hold, where $t_l = w_l - 2p_l$.

Proof. The first equation is a consequence of (10), (11), (5), (13) and Proposition 11, while the second one results from (6), by the use of (9), (5), (13) and Proposition 11.

Lemma 17. *Under hypothesis (H) we have*

$$(29) \quad \begin{aligned} t_p (C_{tqlr} C_{jih}^r - C_{tqjr} C_{lih}^r) = \frac{3-n}{n-2} [(S_{pq} b_t - S_{pt} b_q) C_{hilj} - \\ (S_{ji} b_h - S_{jh} b_i) C_{tqlp} + (S_{li} b_h - S_{lh} b_i) C_{tqjp}] + \\ \frac{1}{n-2} [(-g_{hj} S_{il} + g_{hl} S_{ij} + g_{ij} S_{hl} - g_{il} S_{hj}) b_r C_{pqt}^r + \\ (g_{tp} S_{ql} - g_{tl} S_{qp} - g_{qp} S_{tl} + g_{ql} S_{tp}) b_r C_{jih}^r - \\ (g_{tp} S_{qj} - g_{tj} S_{qp} - g_{qp} S_{tj} + g_{qj} S_{tp}) b_r C_{lih}^r + \\ C_{hpqt} (S_{il} b_j - S_{ij} b_l) - C_{ipqt} (S_{hl} b_j - S_{hj} b_l) - \\ C_{tjih} (S_{ql} b_p - S_{qp} b_l) + C_{qjih} (S_{tl} b_p - S_{tp} b_l) + \\ C_{tilh} (S_{qj} b_p - S_{qp} b_j) - C_{qlih} (S_{tj} b_p - S_{tp} b_j)], \end{aligned}$$

where $t_l = w_l - 2p_l$.

Proof. Transvecting (28) with C_{pqt}^k , in virtue of Lemma 13, we obtain

$$(30) \quad \begin{aligned} t_l C_{hijr} C_{pqt}^r - t_j C_{hilr} C_{pqt}^r + t_r C_{pqt}^r C_{hilj} = \\ \frac{1}{n-2} [C_{hpqt} (S_{il} b_j - S_{ij} b_l) - C_{ipqt} (S_{hl} b_j - S_{hj} b_l) + \\ (-g_{hj} S_{il} + g_{hl} S_{ij} + g_{ij} S_{hl} - g_{il} S_{hj}) b_r C_{pqt}^r]. \end{aligned}$$

Changing in (30) the indices (h, i, j, p, q, t) into (t, q, p, j, i, h) respectively, we have

$$(31) \quad \begin{aligned} t_l C_{tqpr} C_{jih}^r - t_p C_{tqlr} C_{jih}^r + t_r C_{jih}^r C_{tqlp} = \\ \frac{1}{n-2} [C_{tjih} (S_{ql} b_p - S_{qp} b_l) - C_{qjih} (S_{tl} b_p - S_{tp} b_l) + \\ (-g_{tp} S_{ql} + g_{tl} S_{qp} + g_{qp} S_{tl} - g_{ql} S_{tp}) b_r C_{jih}^r], \end{aligned}$$

whence, changing the indices j and l ,

$$(32) \quad \begin{aligned} & t_j C_{tqpr} C_{lih}^r - t_p C_{tqjr} C_{lih}^r + t_r C_{lih}^r C_{tqjp} = \\ & \frac{1}{n-2} [C_{tlih}(S_{qj}b_p - S_{qp}b_j) - C_{qlih}(S_{tj}b_p - S_{tp}b_j) + \\ & (-g_{tp}S_{qj} + g_{tj}S_{qp} + g_{qp}S_{tj} - g_{qj}S_{tp})b_r C_{lih}^r]. \end{aligned}$$

Finally, adding (30) to (32) subtracting (31) and making use of (27) we get (29). This completes the proof.

Lemma 18. *Under hypothesis (H) suppose that $b_l \neq 0$ at a point of M . If $\dim M > 4$ then:*

$$(33) \quad t_r S_p^r = w_r S_p^r = 0,$$

$$(34) \quad \text{rank}[S_{ij}] = 1$$

at the point.

Proof. Transvecting (28) with S_p^l we obtain

$$(35) \quad t_r S_p^r C_{hijk} = \frac{1}{n-2} [S_{hp}(S_{ij}b_k - S_{ik}b_j) - S_{ip}(S_{hj}b_k - S_{hk}b_j)],$$

whence, by transvecting with t^h and the use of (27), we have

$$(36) \quad (n-4)t_r S_p^r (S_{ij}b_k - S_{ik}b_j) = -(t_r S_j^r b_k - t_r S_k^r b_j)S_{ip}.$$

Transvecting (36) with t^i we find

$$(37) \quad (n-3)t_r S_p^r (t_r S_j^r b_k - t_r S_k^r b_j) = 0.$$

If $t_r S_p^r \neq 0$, then (37) and (36) yield $S_{ij}b_k - S_{ik}b_j = 0$, which, in virtue of (35), implies (33), a contradiction. Thus (33) holds good. Now (34) results from (35) and (33). This completes the proof.

Proposition 19. *Suppose that on a manifold M , $\dim M > 4$, hypothesis (H) is satisfied. If $b_l \neq 0$ at a point $x \in M$, then on some neighbourhood of x there exists null (i.e. isotropic) parallel vector field*

$$(38) \quad v_i = \exp[-\frac{1}{2}b] k_i, \quad \partial_i b = b_i,$$

related to the Ricci tensor by

$$(39) \quad S_{ij} = \epsilon k_i k_j, \quad |\epsilon| = 1.$$

Proof. Relation (39) is equivalent to (34). Since the Ricci tensor is recurrent, relations (13) and (39) yield $(k_{i,l} - \frac{1}{2}b_l k_i)k_j + (k_{j,l} - \frac{1}{2}b_l k_j)k_i = 0$. Hence the vector field k is recurrent. By Lemma 13 there exists a function, say b , such that $\partial_i b = b_i$. Moreover k is isotropic by (10) and Proposition 11. Therefore the vector field defined by (38) is isotropic and parallel. This completes the proof.

Lemma 20. *Suppose that on a manifold M , $\dim M > 4$, hypothesis (H) is satisfied. Then*

$$S_{ij}b_r C_{lqt}^r - S_{il}b_r C_{jqt}^r = 0.$$

Proof. We can suppose $b_l \neq 0$. Transvecting (30) with t^h by the use of (27) and (33) we find

$$(n-4) \left[(S_{ij}t_l - S_{il}t_j) b_r C_{pqt}^r - \frac{n-3}{n-2} (S_{ij}b_l - S_{il}b_j) (S_{pq}b_t - S_{pt}b_q) \right] = 0.$$

Alternating the last equation in (p, i) , multiplying the result by k_m and taking into consideration (39) we obtain

$$(t_l k_j - t_j k_l) (S_{im}b_r C_{pqt}^r - S_{pm}b_r C_{iqt}^r) = 0,$$

whence, by transvecting with C_{abc}^l and the use of Lemma 13,

$$k_j t_s C_{abc}^s (S_{im} b_r C_{pqt}^r - S_{pm} b_r C_{iqt}^r) = 0.$$

But $t_s C_{abc}^s = 0$, in virtue of (27) and Lemma 13, implies $S_{ab} b_c C_{def}^c = 0$. Thus the Lemma holds good.

The components $Q(S, C)_{hiqtpj}$ of the tensor field $Q(S, C)$ are given by

$$Q(S, C)_{hiqtpj} = S_{hp} C_{jiqt} - S_{hj} C_{piqt} + S_{ip} C_{hjqt} - S_{ij} C_{hpqt} + \\ S_{qp} C_{hijt} - S_{qj} C_{hipt} + S_{tp} C_{hiqj} - S_{tj} C_{hiqp}.$$

It is well known that if the scalar curvature vanishes and the rank of the Ricci tensor is one, then

$$Q(S, C) = Q(S, R).$$

Lemma 21. *Suppose that on a manifold M , $\dim M > 4$, hypothesis (H) is satisfied. Then*

$$(40) \quad t_r b^r \cdot Q(S, C) = 0.$$

Proof. We can suppose $b_l \neq 0$. Applying Lemma 20 to (29) we obtain

$$(41) \quad \begin{aligned} t_p (C_{tqlr} C_{jih}^r - C_{tqjr} C_{lih}^r) &= \frac{3-n}{n-2} [(S_{pq} b_t - S_{pt} b_q) C_{hilj} - \\ &\quad (S_{ji} b_h - S_{jh} b_i) C_{tqlp} + (S_{li} b_h - S_{lh} b_i) C_{tqjp}] + \\ &\quad \frac{1}{n-2} [(-g_{hj} S_{il} + g_{hl} S_{ij} + g_{ij} S_{hl} - g_{il} S_{hj}) b_r C_{pqt}^r + \\ &\quad (-g_{tl} S_{qp} + g_{ql} S_{tp}) b_r C_{jih}^r - (-g_{tj} S_{qp} + g_{qj} S_{tp}) b_r C_{lih}^r + \\ &\quad C_{hpqt} (S_{il} b_j - S_{ij} b_l) - C_{ipqt} (S_{hl} b_j - S_{hj} b_l) - \\ &\quad C_{tjih} (S_{ql} b_p - S_{qp} b_l) + C_{qjih} (S_{tl} b_p - S_{tp} b_l) + \\ &\quad C_{tlih} (S_{qj} b_p - S_{qp} b_j) - C_{qlih} (S_{tj} b_p - S_{tp} b_j)]. \end{aligned}$$

Transvecting (41) with t^l , making use of (27), (33) and Lemma 13, then alternating the obtained equation in (p, j) and making use of (39) we get (40). This completes the proof.

Lemma 22. *Suppose that on a manifold M , $\dim M > 4$, hypothesis (H) is satisfied and*

$$(42) \quad t_r b^r = 0.$$

Then

$$(43) \quad (n-4) b_r b^r = 0,$$

$$(44) \quad t_l C_{hijk} + t_j C_{hikl} + t_k C_{hilj} = 0,$$

and

$$(45) \quad t_p (C_{tqlr} C_{jih}^r - C_{tqjr} C_{lih}^r) = 0$$

are satisfied.

Proof. Assume $t_r b^r = 0$. Transvecting (41) with t^t , then making use of (27), (33) and Lemma 13 we obtain

$$(46) \quad \begin{aligned} &\frac{n-3}{n-2} [b_h (S_{il} b_j - S_{ij} b_l) - b_i (S_{hl} b_j - S_{hj} b_l)] + \\ &\frac{1}{n-3} (t_j b_r C_{lih}^r - t_l b_r C_{jih}^r) + \\ &\frac{1}{n-2} b_r b^r (-g_{hj} S_{il} + g_{hl} S_{ij} + g_{ij} S_{hl} - g_{il} S_{hj}) = 0, \end{aligned}$$

whence, by transvecting with $b^j b^i$ we find (43).

Transvecting (28) with b^k we easily get

$$t_l b_r C_{jih}^r - t_j b_r C_{lih}^r = \frac{1}{n-2} [b_h (S_{il} b_j - S_{ij} b_l) - b_i (S_{hl} b_j - S_{hj} b_l)]$$

which, combined with (43), (46) and (39) yields

$$(n-4)(S_{al}b_j - S_{aj}b_l)(S_{bh}b_i - S_{bi}b_h) = 0.$$

But, in view of Lemma 13, the last relation gives $b_r C_{jih}^r = 0$. Now (44) and (45) are simply consequences of (28) and (29) respectively. Thus the Lemma is proved.

We end this section with well-known.

Lemma 23 (cf. [Rot82a], [Rot74], Lemma 4 and [Wal50], p. 45). *Suppose that at a point of a manifold M , $\dim M \geq 4$, relations*

$$(47) \quad Q(S, C) = 0,$$

$$S_{ij} = \epsilon k_i k_j, \quad |\epsilon| = 1,$$

and

$$r = 0$$

are satisfied. Then

$$(48) \quad k_l C_{hijk} + k_j C_{hikl} + k_k C_{hilj} = 0$$

and

$$(49) \quad R_{hijk} = k_i k_j T_{hk} - k_i k_k T_{hj} + k_h k_k T_{ij} - k_h k_j T_{ik},$$

where $T_{ij} = z^r z^s R_{rij s}$, $z^r k_r = 1$.

3. MAIN RESULTS

Proposition 24. *Suppose that on a manifold M , $\dim M > 4$, hypothesis (H) is satisfied. If at $x \in M$, $t_r b^r(x) \neq 0$ (hence $b_l(x) \neq 0$), then on some neighbourhood of x the Riemann-Christoffel curvature tensor has the form (49).*

Proof. By (39), (40), (10) and Proposition 11 the assumption of Lemma 23 are satisfied on some neighbourhood of x . This completes the proof.

Corollary 25. *There do not exist quasi-recurrent manifolds which are Ricci-flat.*

Proof. On arbitrary quasi-recurrent and Ricci-flat manifold M relation (9) holds. Suppose that the curvature tensor of M does not vanish on an open subset $U \subset M$. Let V_r , $r = 1, \dots, n$ be a family of non-conformally flat manifolds with recurrent curvature tensor and nowhere vanishing Ricci tensors indexed by r . Then the product manifold $U \times (\times_{r=1}^n V_r)$, $n > 1$, would be a conformally quasi-recurrent and Ricci-recurrent manifold and $\text{rank}[S_{ij}] \geq 2$, a contradiction.

As a consequence of Propositions 11, 19 and 24 we obtain

Corollary 26. *Suppose that on a manifold M , $\dim M > 4$, hypothesis (H) and $Q(S, C) = 0$ hold. If $b_l \neq 0$ at a point $x \in M$, then on some neighbourhood of x the metric g is of the Walker type. Moreover, M is semi-symmetric, i.e. $R \cdot R = 0$.*

On the other hand, if $Q(S, C) \neq 0$, then (45), Lemma 13 and Proposition 11 result in

Corollary 27. *Let on a manifold M , $\dim M > 4$, hypothesis (H) be satisfied. Suppose moreover that $Q(S, C) \neq 0$ and $w - 2p \neq 0$. Then*

$$R \cdot R = Q(S, R).$$

In virtue of Proposition 15 we conclude with the following

Theorem 28. *Let M , $\dim M > 4$, be a manifold with nowhere vanishing Weyl conformal curvature tensor and Ricci tensor.*

Suppose that M is conformally quasi-recurrent with recurrent but non-parallel Ricci tensor and, moreover, is non-conformally related to a conformally symmetric manifold.

Then M must be necessary Ricci-generalized pseudosymmetric manifold.

4. REMARKS ON THE EXISTENCE

There exist Ricci-generalized pseudosymmetric manifolds which are neither conformally quasi-recurrent nor Ricci-recurrent in an essential way.

Example. Let M , $\dim M > 4$, be a Ricci-recurrent manifold which is simultaneously conformally recurrent. Then from results of ([Rot82a], [Rot74], see also [EK91]) it follows that on a neighbourhood of a generic point relations (2) and $R \cdot R = Q(S, R) = 0$ are satisfied.

Example. The warped product $M_1 \times_F M_2$ of a 2-dimensional manifold $(M_1, \bar{g})$ and a manifold of constant curvature $(M_2, \tilde{g})$, $\dim M_2 \geq 2$, is a manifold satisfying $R \cdot R = Q(S, R)$ under some metric condition imposed on F ([DD91b], Corollary 5.1), however it could not be a non-conformally recurrent conformally quasi-recurrent and Ricci-recurrent one.

Example. Let $(M, \tilde{g})$ be a hypersurface of a semi-Riemannian manifold (N, g) of constant curvature K , $\dim N = n + 1$, $n \geq 3$, $\tilde{g}$ being the metric tensor of M induced from g . Then the condition

$$\tilde{R} \cdot \tilde{R} = Q(\tilde{S}, \tilde{R}) - \frac{n-2}{n(n+1)} K Q(\tilde{g}, \tilde{C})$$

holds on $(M, \tilde{g})$ ([DV91]).

It is clear that $\tilde{R} \cdot \tilde{R} = Q(\tilde{S}, \tilde{R})$ implies $K Q(\tilde{g}, \tilde{C}) = 0$, whence $K \tilde{C} = 0$ results. Thus $K = 0$ is the only case when hypersurfaces satisfying the required condition can exist.

Example. The warped product $M_1 \times_F M_2$, where M_1 is an open interval in R , M_2 is a locally symmetric Einstein space of non-constant curvature and $F = p^2(x^1)$, $p''(x^1) \neq 0$,

$$\tilde{K} + (n-1)(n-2)(p'(x^1))^2 + (n-1)p(x^1)p''(x^1) = 0,$$

$\tilde{K}$ being the scalar curvature of M_2 , is a non-conformally recurrent (hence non-conformally flat) conformally quasi-recurrent manifold. The fundamental forms w , p satisfy $w = 2p$ and $dp \neq 0$. Hence, in virtue of Corollary 14, the manifold is not Ricci-recurrent.

REFERENCES

- [AM67] T. Adati and T. Miyazawa. On a space with recurrent conformal curvature. *Tensor (N.S.)*, 18:348–354, 1967.
- [CC69] M. C. Chaki and A. N. Roy Chowdhury. On conformally recurrent spaces of second order. *J. Australian Math. Soc.*, 10:155–161, 1969.
- [Cha87] M. C. Chaki. On pseudosymmetric manifolds. *Analele Ştiinţifice Ale Universităţii "Al. I. Cuza" Iaşi*, 33:53–58, 1987.
- [Cha94] M. C. Chaki. On generalized pseudosymmetric manifolds. *Publ. Math. Debrecen*, 45:305–312, 1994.
- [CM97] M. C. Chaki and S. P. Mondal. On generalized pseudo symmetric manifolds. *Publ. Math. Debrecen*, 51(1–2):35–42, 1997.
- [CS73] C.D. Collinson and F. Söler. Second order conformally recurrent and birecurrent plane fronted waves. *Tensor, N.S.*, 27:37–40, 1973.
- [DD91a] F. Defever and R. Deszcz. On semi-Riemannian manifolds satisfying the condition $r \cdot r = q(s, r)$. *Geometry and Topology of Submanifolds*, 3:131–147, 1991.
- [DD91b] F. Defever and R. Deszcz. On warped product manifolds satisfying a certain curvature condition. *Atti Accad. Peloritana Pericolanti CL. Sci. Fis. Mat. Natur.*, 61:213–236, 1991.
- [De00] U. C. De. On conformally flat pseudosymmetric spaces. *Balkan Journal of Geometry and its Applications*, 5(2):61–64, 2000.
- [DV91] R. Deszcz and L. Verstraelen. Hypersurfaces of semi-Riemannian conformally flat manifolds. *Geometry and Topology of Submanifolds*, 3:108–130, 1991.

- [Eis38] L. Eisenhart. Fields of parallel vectors in Riemannian space. *Ann. Math.*, 39:316–321, 1938.
- [EK91] S. Ewert-Krzemieniewski. On conformally birecurrent Ricci-recurrent manifolds. *Colloq. Math.*, 62:299–312, 1991.
- [EK92] S. Ewert-Krzemieniewski. On the existence of certain types of Riemannian metrics. *Demonstratio Math.*, 25:851–858, 1992.
- [EK93] S. Ewert-Krzemieniewski. On some generalisation of recurrent manifolds. *Mathematica Pannonica*, 4(2):191–203, 1993.
- [EK98] S. Ewert-Krzemieniewski. On manifolds with curvature condition of recurrent type of the second order. *Periodica Mathematica Hungarica*, 34(2), 1998.
- [GH82] W. Grycak and M. Hotłoś. On the existence of certain types of Riemannian metrics. *Colloq. Math.*, 47:31–37, 1982.
- [Kru61] G. J. Kruchkovich. *V. F. Kagan, Subproektivnye prostranstva*, chapter O prostranstvach V. F. Kagana. Gosudarstvennoe Izdatelstvo Fiziko-Matematicheskoi Literatury, Moskva, 1961. in Russian.
- [Ols87] Z. Olszak. On Ricci-recurrent manifolds. *Colloq. Math.*, 52:205–211, 1987.
- [Pat81] E. M. Patterson. A class of critical Riemannian metrics. *J. London Math. Soc.*, 23(2):349–358, 1981.
- [Prv88] M. Prvanović. Conformally quasi-recurrent manifolds, Finsler and Lagrange spaces. In *Proc. 5th National Seminar on Finsler Spaces, Hon. 60th Birthday R. Miron*, pages 321–332, Braşov, 1988.
- [Rot74] W. Roter. On conformally symmetric Ricci-recurrent spaces. *Colloquium Mathematicum*, 31:87–96, 1974.
- [Rot82a] W. Roter. On conformally recurrent Ricci-recurrent manifolds. *Colloquium Mathematicum*, 46:45–57, 1982.
- [Rot82b] W. Roter. On conformally related conformally recurrent metrics i. Some general results. *Colloquium Mathematicum*, 47:39–46, 1982.
- [Rot87] W. Roter. On the existence of certain conformally recurrent metrics. *Colloq. Math.*, 51:315–327, 1987.
- [SC67] R. N. Sen and M. C. Chaki. On curvature restrictions of certain kind of conformally flat Riemannian spaces of class one. *Proc. Nat. Inst. Sci. India*, 33(A):100–102, 1967.
- [TB92] L. Tamássy and T. Q. Binh. On weakly symmetric and weakly projective symmetric Riemannian manifolds. In *Differential geometry and its applications (Eger, 1989)*, volume 56 of *Colloq. Math. Soc. János Bolyai*, pages 663–670. North-Holland, 1992.
- [Wal50] A.G. Walker. On Ruse’s spaces of recurrent curvature. *Proc. Lond. Math. Soc.*, 52:36–64, 1950.

Received December 17, 2002.

INSTITUTE OF MATHEMATICS,
 TECHNICAL UNIVERSITY OF SZCZECIN,
 AL. PIASTÓW 17, 70-310 SZCZECIN,
 POLAND
E-mail address: ewert@ps.pl