

ON THE CURVATURES OF $(r + 1)$ -DIMENSIONAL GENERALIZED TIME-LIKE RULED SURFACE

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ABSTRACT. In this paper, we obtain some relationships between the curvatures of $(r + 1)$ -dimensional generalized time-like ruled surfaces. We also calculate the drall of a generalized time-like ruled surfaces when the base curve is taken as an orthogonal trajectory of the generated spaces.

1. INTRODUCTION

Let R_1^n be n -dimensional Minkowski space with the standard metric given by

$$\langle, \rangle = -dx_0^2 + dx_1^2 + dx_2^2 + \cdots + dx_{n-1}^2$$

where $(x_0, x_1, \dots, x_{n-1})$ is a rectangular system of R_1^n [2]. Nonzero vectors are classified as time-like, space-like or null, respectively, according to whether

$$\langle v, v \rangle < 0, \quad \langle v, v \rangle > 0, \quad \langle v, v \rangle = 0.$$

Let $\alpha \in R_1^n$ is a curve in Minkowski space. If $\dot{\alpha}$ is a velocity of α and $\langle \dot{\alpha}, \dot{\alpha} \rangle < 0$ then the curve α is called a time-like curve.

Let R_1^n be n -dimensional Minkowski space and M a submanifold of R_1^n . Let $\bar{D}$ denote the standard Riemannian connection of R_1^n and let D denote the Riemannian connection of M . For any vector fields X, Y on M we have the Gauss equation [3].

$$(1.1) \quad \bar{D}_X Y = D_X Y + V(X, Y)$$

where $D_X Y, V(X, Y)$ are tangential and normal components of $\bar{D}_X Y$, respectively. V is called the second fundamental form of M . We also have the Weingarten equation giving the tangential and normal components of $\bar{D}_X \xi$, where ξ is a normal vector field of M :

$$(1.2) \quad \bar{D}_X \xi = -A_\xi(X) + D_X^\perp \xi.$$

Let X, Y be vector fields on M , ξ be a normal vector field and $\langle, \rangle$ be the Minkowski metric on R_1^n . From (1.1) we have

$$(1.3) \quad \langle \bar{D}_X Y, \xi \rangle = \langle V(X, Y), \xi \rangle$$

and then (1.2) implies

$$(1.4) \quad \langle V(X, Y), \xi \rangle = \langle A_\xi(X), Y \rangle.$$

Let $\{\xi_1, \xi_2, \dots, \xi_{n-m}\}$ be an orthonormal basis of $\chi^\perp(M)$ and the space of normal vector fields on M . Then there exist smooth functions $V^j(X, Y)$, $j = 1, \dots, n - m$,

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from M into $\mathbb{R}$ such that

$$(1.5) \quad V(X, Y) = \sum_{j=1}^{n-m} V^j(X, Y) \xi_j.$$

Furthermore we may define the mean curvature vector field H by

$$(1.6) \quad H = \sum_{j=1}^{n-m} \frac{\text{trace } A_{\xi_j}}{m} \xi_j$$

and the mean curvature function as $|H|$. At a point $p \in M$, $H(p)$ is called the mean curvature vector and $|H(p)|$ the mean curvature at p .

If $H(p) = 0$ for each $p \in M$, then M is said to be minimal.

Let I be an open interval and $\alpha: I \rightarrow R_1^n$ be a time-like curve in Minkowski space. For each $t \in I$, let $\{e_1(t), e_2(t), \dots, e_r(t)\}$, $(1 \leq r \leq n-2)$ be an orthonormal set of vectors spanning the r -dimensional space-like subspace $W_r(t)$ of $T_{\alpha(t)}R_1^n$. We have

$$(1.7) \quad \langle e_i, e_j \rangle = \delta_{ij}, \quad (i, j = 1, \dots, r)$$

and denoting by $\dot{e}_i$ the derivative of the vector field e_i along the time-like curve α ;

$$(1.8) \quad \langle \dot{e}_i, e_j \rangle + \langle e_i, \dot{e}_j \rangle = 0.$$

We may then define an $(r+1)$ -dimensional submanifold of R_1^n as in [5].

Definition 1.1. Let $\alpha, \{e_i\}$ be as above and define $\varphi: I \times R_1^r \subseteq R_1^n$ by

$$(1.9) \quad \varphi(t, u_1, \dots, u_r) = \alpha(t) + \sum_{i=1}^r u_i e_i(t)$$

for all $(t, u_1, \dots, u_r) \in I \times R_1^r$. Let $M = \varphi(G)$ where $G = I \times R_1^r \subseteq R_1^{r+1}$. Note that

$$\text{rank}(\varphi_t, \varphi_{u_1}, \dots, \varphi_{u_r}) = \text{rank} \left(\dot{\alpha}(t) + \sum_{i=1}^r u_i \dot{e}_i(t), e_1(t), e_2(t), \dots, e_r(t) \right) = r+1$$

so M is an $(r+1)$ -dimensional submanifold of R_1^n . We call M an $(r+1)$ -dimensional generalized time-like ruled surface. The time-like curve α is called the *base curve* of the generalized time-like ruled surface and the space-like subspace $W_r(t)$ is called the *generating space* (or briefly, *generation*) at the point $\alpha(t)$.

Definition 1.2. The subspace $A(t)$ given by

$$(1.10) \quad A(t) = \text{Span}\{e_1(t), e_2(t), \dots, e_r(t), \dot{e}_1(t), \dot{e}_2(t), \dots, \dot{e}_r(t)\}$$

with dimension $\dim A(t) = r+m$, $0 \leq m \leq r$, is said to be the asymptotic bundle of the generalized time-like ruled surface.

$W_r(t)$ is a subspace of $A(t)$ and using the Gramm–Schmidt orthogonalization process, basis of the form;

$$(1.11) \quad \{e_1(t), e_2(t), \dots, e_r(t), a_{r+1}(t), \dots, a_{r+m}(t)\}$$

may be found. Then there exist b_{ij}, c_{ik} such that

$$(1.12) \quad \dot{e}_i = \sum_{j=1}^r b_{ij} e_j + \sum_{k=1}^m c_{ik} a_{r+k}$$

with $b_{ij} = -b_{ji}$ by (1.8). The basis $\{e_1(t), e_2(t), \dots, e_r(t)\}$ is called the *natural carrier basis* of $W_r(t)$.

Now let $\tau_{m+1} = \langle \dot{\alpha}_r(t), a_{r+m+1} \rangle$, and $K_k = \langle \dot{e}_k(t), a_{r+k} \rangle$ for $k = 1, \dots, m$ so that

$$\dot{e}_i = \sum_{j=1}^r b_{ij} e_j + K_i a_{r+i}, \quad (1 \leq i \leq m, K_i > 0)$$

$$\dot{e}_i = \sum_{j=1}^r b_{ij} e_j \quad (m < i \leq r) \quad [1].$$

We now define the following:

$$(1.13) \quad \delta_k = \frac{\tau_{m+1}}{K_k}, \quad k = 1, \dots, m$$

and note that each δ_k is invariant under a reparametrization $t \rightarrow t^*$ with $\frac{dt}{dt^*} > 0$. δ_k is called the k th principle drall (principal distribution parameter) of M lying in $W_r(t)$. The drall (distribution parameter) of M is defined by

$$(1.14) \quad \delta = |\delta_1 \dots \delta_m|^{\frac{1}{m}}$$

as in [1].

2. ON THE CURVATURES OF GENERALIZED TIME-LIKE SURFACES

Let M be an $(r+1)$ -dimensional generalized time-like ruled surface and s the arc length parameter of the time-like curve α . Let $\{e_1(s), e_2(s), \dots, e_r(s)\}$ be an orthonormal basis of the generating space-like space $W_r(s)$. Let us choose the base time-like curve α to be an orthogonal trajectory of the generating spaces $W_r(s)$. M is given by

$$(2.1) \quad \varphi(s, u_1, \dots, u_r) = \alpha(s) + \sum_{i=1}^r u_i e_i(s), \quad u_i \in R.$$

Let $\{e_0, e_1, \dots, e_r\}$ be a (local) orthonormal basis of the space of vector fields $\chi(M)$ and let us choose $e_0 = \varphi_*(\frac{\partial}{\partial s})$. By (2.1)

$$(2.2) \quad \varphi_s = \dot{\alpha}(s) + \sum_{i=1}^r u_i \dot{e}_i(s), \quad \varphi_{u_i} = e_i(s)$$

then

$$(2.3) \quad \overline{D}_{e_i} e_j = 0, \quad i, j = 1, \dots, r$$

and using (1.1),

$$(2.4) \quad V(e_i, e_j) = 0, \quad i, j = 1, \dots, r.$$

Since $\overline{D}_{e_i} e_0 \perp e_j$ and $\overline{D}_{e_i} e_0 \perp e_0$ (for all i, j) then

$$(2.5) \quad \overline{D}_{e_i} e_0 = V(e_i, e_0), \quad i, j = 1, \dots, r.$$

Let $\{\xi_1, \dots, \xi_{n-r-1}\}$ be an orthonormal basis of normal vector fields. Then $\{e_0, e_1, \dots, e_r, \xi_1, \dots, \xi_{n-r-1}\}$ gives a basis of $T_p R_1^n$ for each point $p \in M$. Let us write

$$(2.6) \quad \begin{aligned} \overline{D}_{e_0} \xi_j &= a_{00}^j e_0 + \sum_{t=1}^r a_{0t}^j e_t + \sum_{q=1}^{n-r-1} b_{0q}^j \xi_q, \quad j = 1, \dots, n-r-1 \\ \overline{D}_{e_i} \xi_j &= a_{0i}^j e_0 + \sum_{t=1}^r a_{it}^j e_t + \sum_{q=1}^{n-r-1} b_{iq}^j \xi_q, \quad i = 1, \dots, r \end{aligned}$$

where the a_{it}^j are coefficients of the matrix of A_{ξ_j} :

$$(2.7) \quad A_{\xi_j} = - \begin{bmatrix} a_{00}^j & a_{01}^j & \cdots & a_{0r}^j \\ a_{01}^j & a_{11}^j & \cdots & a_{1r}^j \\ \vdots & \vdots & \ddots & \vdots \\ a_{0r}^j & a_{r1}^j & \cdots & a_{rr}^j \end{bmatrix}, \quad (j = 1, \dots, n-r-1).$$

We simplify this matrix using (2.6),

$$\langle \overline{D}_{e_i} e_t, \xi_j \rangle = -a_{it}^j, \quad (i, t = 1, \dots, r; \quad j = 1, \dots, n-r-1)$$

and by (2.3), $a_{it}^j = 0$. (2.7) can be written as follows

$$(2.8) \quad A_{\xi_j} = - \begin{bmatrix} a_{00}^j & a_{01}^j & \cdots & a_{0r}^j \\ a_{01}^j & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{0r}^j & 0 & \cdots & 0 \end{bmatrix}.$$

Furthermore, (2.6) and (1.4) respectively lead to the relations:

$$\langle \overline{D}_{e_i} e_0, \xi_j \rangle = -a_{0i}^j, \quad (i = 1, \dots, r; \quad j = 1, \dots, n-r-1)$$

and

$$\langle V(e_i, e_0), \xi_j \rangle = \langle A_{\xi_j}(e_i), e_0 \rangle = a_{0i}^j, \quad (1 \leq i \leq r, \quad 1 \leq j \leq n-r-1)$$

and therefore by (1.5) and (2.5)

$$(2.9) \quad V(e_i, e_0) = \overline{D}_{e_i} e_0 = \sum_{j=1}^{n-r-1} a_{0i}^j \xi_j, \quad (j = 1, \dots, r).$$

Now, let X, Y be vector fields on the m -dimensional submanifold M whose curvature tensor field is R . As in [4], we have

$$(2.10) \quad \langle X, R(X, Y)Y \rangle = \langle V(X, X), V(Y, Y) \rangle - \langle V(X, Y), V(X, Y) \rangle$$

where V is 2nd fundamental form of M embedded in R_1^n .

Definition 2.1. Let M be any m -dimensional Lorentzian manifold with curvature tensor R and $\{e_1, \dots, e_m\}$ be an orthonormal basis of $T_p M$ for $p \in M$. Then the *Ricci curvature tensor field* Ric is defined by

$$(2.11) \quad \text{Ric}(X, Y) = \sum_{i=1}^m \varepsilon_i \langle R(e_i, Y)X, e_i \rangle$$

where

$$\varepsilon_i = \begin{cases} -1, & i = 1 \\ 1, & 2 \leq i \leq m. \end{cases}$$

The *scalar curvature* of M is defined by

$$(2.12) \quad \mathbf{r} = \sum_{i=1}^m \varepsilon_i \text{Ric}(e_i, e_i)$$

or by (2.11)

$$(2.13) \quad \mathbf{r} = \sum_{i=1}^m \sum_{j=1}^m \varepsilon_i \varepsilon_j \langle R(e_j, e_i)e_i, e_j \rangle.$$

In order to calculate the Ricci curvature of M in the direction of the vector fields e_t ($t = 1, \dots, r$), we use (2.4), (2.9), (2.10), (2.11) and we obtain

$$(2.14) \quad \text{Ric}(e_t, e_t) = \sum_{j=1}^{n-r-1} (a_{0t}^j)^2, \quad (t = 1, \dots, r),$$

and

$$(2.15) \quad \text{Ric}(e_0, e_0) = \sum_{t=1}^r \sum_{j=1}^{n-r-1} (a_{0t}^j)^2.$$

From (2.14) and (2.15) we can define

$$(2.16) \quad \text{Ric}(e_0, e_0) = \sum_{t=1}^r \text{Ric}(e_t, e_t).$$

Now we have proved the following theorem.

Theorem 2.1. *Let $\{e_1, \dots, e_r\}$ be an orthonormal basis of the generating space of the $(r+1)$ -dimensional generalized time-like ruled surface M and $\{e_0, e_1, \dots, e_r\}$ and orthonormal basis of $\chi(M)$. If the base time-like curve of M is chosen as an orthonormal trajectory of generating space then the Ricci curvature in the direction of e_0 is equal to the sum Ricci curvatures in the directions of the vector fields forming a basis of the generating space.*

By (2.12), the scalar curvature of the $(r+1)$ -dimensional generalized time-like ruled surface M may be expressed as

$$\begin{aligned} \mathbf{r} &= -\text{Ric}(e_0, e_0) + \sum_{t=1}^r \text{Ric}(e_t, e_t), \\ \mathbf{r} &= 0. \end{aligned}$$

Using (1.3), (1.4), (1.5), (2.6) and (2.8) we have

$$(2.17) \quad V(e_0, e_0) = - \sum_{i=1}^{n-r-1} (\text{trace } A_{\xi_j}) \xi_j.$$

Now (1.6) gives

$$H = -\frac{1}{r+1} V(e_0, e_0).$$

If M is minimal then H is zero and so

$$(2.18) \quad V(e_0, e_0) = 0.$$

We say that $X_p, Y_p \in T_p M$ are conjugate if $V(X_p, Y_p) = 0$. We have the following theorem.

Theorem 2.2. *Let $\{e_1, \dots, e_r\}$ be an orthonormal basis for the generating space of an $(r+1)$ -dimensional generalized time-like ruled surface M and let e_0 be an unit time-like tangent vector field to the base time-like curve, the latter taken to be an orthonormal trajectory of the generating space of M . Then the minimal time-like ruled surface M is totally geodesic iff e_0 is conjugate to each vector e_i , $i = 1, \dots, r$.*

Proof. $\{e_0, e_1, \dots, e_r\}$ is an orthonormal basis of $\chi(M)$ and for each $X, Y \in \chi(M)$ we may write

$$X = a_0 e_0 + \sum_{i=1}^r a_i e_i, \quad Y = b_0 e_0 + \sum_{i=1}^r b_i e_i$$

and then

$$(2.19) \quad V(X, Y) = a_0 b_0 V(e_0, e_0) + \sum_{i=1}^r (a_i b_0 + a_0 b_i) V(e_i, e_0) + \sum_{i=1}^r a_i b_i V(e_i, e_i).$$

($\implies$) If M is totally geodesic then V is identically zero, so e_0 is certainly conjugate to $e_i, i = 1, \dots, r$.

($\impliedby$) $V(e_0, e_i) = 0$ for $i = 1, \dots, r$ then by (2.4) and (2.18), (2.19) reduces to $V(X, Y) = 0$ and this completes the proof of the theorem. $\square$

If trace $A_{\xi_j} = -a_{00}^j$ from (2.8) is substituted into (1.6) we obtain

$$(2.20) \quad (r+1)^2 |H|^2 = \sum_{j=1}^{n-r-1} (a_{00}^j)^2.$$

Theorem 2.3. *Let M be an $(r+1)$ -dimensional generalized time-like ruled surface. Let the base time-like curve α be an orthogonal trajectory of the generating space be parametrized by arc length. Then k th principle distribution parameter is*

$$\delta_k = \frac{(1 - \sum_{t=1}^r \tau_t^2)^{\frac{1}{2}}}{\left| \|\overline{D}_{e_0} e_k\|^2 - \sum_{j=1}^r \langle \overline{D}_{e_0} e_k, e_j \rangle^2 \right|^{\frac{1}{2}}}$$

($k = 1, \dots, m$) and the distribution parameter (drall) is

$$\delta_k = \frac{(1 - \sum_{t=1}^r \tau_t^2)^{\frac{1}{2}}}{\prod_{k=1}^m \left| \|\overline{D}_{e_0} e_k\|^2 - \sum_{j=1}^r \langle \overline{D}_{e_0} e_k, e_j \rangle^2 \right|^{\frac{1}{2m}}}.$$

Proof. Using (1.13) and (1.14) we obtain

$$(2.21) \quad \delta_k = \frac{\left\| \dot{\alpha} - \sum_{j=1}^r \langle \dot{\alpha}, e_j \rangle e_j - \sum_{t=1}^r \langle \dot{\alpha}, a_{r+t} \rangle a_{r+t} \right\|}{\left\| \dot{e}_k - \sum_{j=1}^r \langle \dot{e}_k, e_j \rangle e_j \right\|}.$$

The base curve α is an orthogonal trajectory so $\langle \dot{\alpha}, e_j \rangle = 0$ for $j = 1, \dots, r$. Substituting $\langle \dot{\alpha}, e_j \rangle = 0$, ($j = 1, \dots, r$) $\dot{e}_k = \overline{D}_{e_0} e_k$, and $\langle \dot{\alpha}, a_{r+t} \rangle = \tau_t$, ($t = 1, \dots, m$) into (2.21), the desired result is obtained. $\square$

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