

ε -ISOMETRIC APPROXIMATION PROBLEM

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ABSTRACT. In this paper, some problems for isometric approximation is resolved.

1. INTRODUCTION

Let E and F be normed linear spaces. Hyers and Ulam [6] called the mapping $T: E \rightarrow F$ an absolute error ε -isometry if for any $\varepsilon \geq 0$,

$$(1) \quad \|x - y\| - \varepsilon \leq \|Tx - Ty\| \leq \|x - y\| + \varepsilon$$

for any $x, y \in E$. On the stability of isometry, Hyers and Ulam asked following questions:

1. For each *surjective* ε -isometry T , if there exists an isometric mapping $U: E \rightarrow F$, and a constant K such that

$$\|Tx - Ux\| \leq K(E, F)\varepsilon$$

for any $x \in E$ where the constant K depends only on E and F .

2. If the answer above is positive, what is the best K ?

To start with studying these problems, without loss of generality, $T(0) = 0$ for T is ε -isometry, $T - T(0)$ is necessary ε -isometry. P.M. Grubern [4] in 1978, T.M. Rassias and P. Šemel [12] in 1993 gave that the positive answer.

The ε -isometry $T: E \rightarrow F$ is called Lipschitz ε -isometry if

$$(2) \quad (1 - \varepsilon)\|x - y\| \leq \|Tx - Ty\| \leq (1 + \varepsilon)\|x - y\|$$

for all $x, y \in E$.

Now, suppose that Lipschitz ε -isometry T is a linear operator, Benyamini [2], Alspach [1] and Dingguangui [5] proved that there exists an isometric approximation of T . When T is nonlinear and surjective operator, K. Jarosz [7] obtained positive answer on $C_0(X) \rightarrow C_0(Y)$, where X, Y are locally compact Hausdorff spaces.

Withdrawing the condition of *surjective* and *linear*, how about Lipschitz ε -isometric approximation problem? G.M. Lövlom [9, 10] gave two local results for these problems, i.e. to restrict the problem on the unit ball $B_1(C(X)) \rightarrow B_1(C(Y))$ where X, Y are compact Hausdorff spaces, the answer is positive. Two counterexamples given show that as $E = F = l_1$ or $E = F = (L_1(0, 1) \times R)_1$ the local problem is negative.

In this paper we restrict ourselves to the local question about absolute error ε -isometry (1) without the assumption of *surjective* and we have some changed for the definition of T as follows.

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$T: E \rightarrow F$ is an ε -isometry, meaning that

$$(3) \quad \|x - y\| - \varepsilon \leq \|Tx - Ty\| \leq \|x - y\|$$

for any $x, y \in E$.

Thanks to Löfblom's idea, we prove that the ε -isometric problem (3) on

$$B_1(C(X)) \rightarrow B_1(C(Y))$$

is positive, and on $B_1(E) \rightarrow B_1(F)$ where $E = F = l_1$ or $E = F = (L_1(0, 1) \times R)$ the problem is negative.

2. ε -ISOMETRY ON $B_1(C(X)) \rightarrow B_1(C(Y))$

Let X, Y be compact metric spaces with metrics d_1 and d_2 and let $B_R(C(X))$ denote the ball of $C(X)$ with center 0 and radius R .

Theorem 2.1. *Let $T: B_1(C(X)) \rightarrow B_1(C(Y))$ with $T(0) = 0$, and*

$$(4) \quad \|f - g\| - \varepsilon \leq \|Tf - Tg\| \leq \|f - g\|$$

for any $f, g \in B_1(C(X))$. Then there exists an isometry

$$U: B_{1-\delta_1(\varepsilon)}(C(X)) \rightarrow B_1(C(Y))$$

such that

$$\|Tf - Uf\| \leq \varepsilon$$

on $B_{1-\delta_1(\varepsilon)}(C(X))$, where $\delta_1(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$.

The proof is based on the following Lemmas. Let a be fixed, $4\varepsilon < a \leq 1$.

Definition 2.2 ([9]). Given $x_0 \in X$, we say that $f \in C(X)$ is a tentfunction at x_0 if for some $\delta > 0$

$$(5) \quad f(x) = \begin{cases} 1 - \frac{d_1(x_0, x)}{\delta}, & x \in B(x_0, \delta), \\ 0, & \text{otherwise.} \end{cases}$$

obviously, $f(x_0) = 1, \|f\| = 1$.

Lemma 2.3. *Let $\{f_n\} \subset B_1(C(X))$, $\{x_n\} \subset X$, $\{y_n\} \subset Y$ be sequences with $y_n \rightarrow y$ and f_n a tentfunction at x_n with $\text{supp}(f_n) = B(x_n, \delta_n)$ where $\delta_n \rightarrow 0$ when $n \rightarrow \infty$.*

If for all n

$$(6) \quad 2a - \varepsilon \leq |T(af_n)(y_n) - T(-af_n)(y_n)|,$$

then $\lim_{n \rightarrow \infty} x_n$ exists.

Proof. X is a compact metric space, so $\{x_n\}$ contains a convergent subsequence, say $\{x_{n'}\}$ with $\lim_{n' \rightarrow \infty} x_{n'} = x$. Assume that x_n is not convergent. Then for some $d > 0$, there exists, for every N , $n \geq N$ such that $d_1(x_n, x) \geq d$. Let $g \in C(X)$ with $0 \leq g \leq \frac{a}{2}$, $g = \frac{a}{2}$ on $B(x, \frac{d}{4})$ and with $\text{supp}(g) \subset B(x, \frac{d}{2})$.

For each N it is possible to find $n, n' \geq N$ such that $\text{supp}(f_{n'}) \subset B(x, \frac{d}{4})$ and $B(x_n, \delta_n) \cap B(x, \frac{d}{2}) = \emptyset$. Then we have

$$(7) \quad \|g - af_n\| = \frac{a}{2}, \|g + af_n\| = \frac{3a}{2}, \|g \pm af_n\| = a.$$

Because T is ε -isometry, therefore for any $y \in Y, f, g \in B_1(C(X))$

$$|T(g)(y) - T(f)(y)| \leq \|g - f\|$$

Thus

$$-\|g - f\| + T(f)(y) \leq T(g)(y) \leq \|g - f\| + T(f)(y).$$

We get

$$(8) \quad T(af_{n'})(y_{n'}) - \frac{a}{2} \leq T(g)(y_{n'}) \leq T(af_{n'})(y_{n'}) + \frac{a}{2}.$$

$$(9) \quad T(-af_n)(y_n) - \frac{3a}{2} \leq T(g)(y_n) \leq T(-af_n)(y_n) + \frac{3a}{2}.$$

$$(10) \quad T(\pm af_n)(y_n) - a \leq T(g)(y_n) \leq T(\pm af_n)(y_n) + a.$$

By hypothesis of T with $T(0) = 0$, we have for all n .

$$(11) \quad \|T(\pm af_n)\| \leq a.$$

$$(12) \quad \begin{cases} T(af_n)(y_n) \geq T(-af_n)(y_n) + 2a - \varepsilon, \\ T(af_n)(y_n) \leq T(-af_n)(y_n) - 2a + \varepsilon. \end{cases}$$

From (8)–(12) we get

$$(13) \quad \begin{cases} a - \varepsilon \leq T(af_n)(y_n) \leq a, \\ -a \leq T(-af_n)(y_n) \leq -a + \varepsilon. \end{cases}$$

$$(14) \quad \begin{cases} -a \leq T(af_n)(y_n) \leq -a + \varepsilon, \\ a - \varepsilon \leq T(-af_n)(y_n) \leq a. \end{cases}$$

By (12) and (14) we obtain that

$$\pm \frac{a}{2} - \varepsilon \leq T(g)(y_{n'}) \leq \pm \frac{a}{2} + \varepsilon.$$

Thus we have

$$|T(g)(y_{n'})| \geq \frac{a}{2} - \varepsilon > \varepsilon \text{ and } |T(g)(y_n)| \leq \varepsilon.$$

Since $T(g) \in C(Y)$, $4\varepsilon < a \leq 1$ fixed and $d_2(y_{n'}, y_n) \rightarrow 0$ when $n, n' \rightarrow \infty$, this clearly gives a contradiction for n, n' large enough. Hence $\{x_n\}$ is convergent. $\square$

Definition 2.4 ([9]). We say $y \in A_x$ if there exist sequences $\{f_n\}, \{x_n\}, \{y_n\}$ satisfying the conditions in Lemma 2.3 with $x = \lim x_n$ and $y = \lim y_n$.

Lemma 2.5. The set $\bigcup_{x \in X} A_x$ is closed and mapping

$$\varphi: \bigcup_{x \in X} A_x \rightarrow X, \quad \varphi(y) = x, \quad y \in A_x$$

is well-defined and continuous.

Proof. The proof of Lemma is same as G.M. Lövblom's [9] although the two definitions of isometry is different. $\square$

Lemma 2.6. Let $y \in A_x$ and let $\{f_{kn}\}, \{x_{kn}\}$ and $\{y_{kn}\}$ be any collection of sequences satisfying the conditions in Lemma 2.3. Then

$$\lim_{n \rightarrow \infty} \text{sign } T(af_n)(y_n) = \text{sign } T\left(\frac{a}{2}\right)(y).$$

Proof. For each y_n we have $\text{sign } T(af_n)(y_n) = \text{sign } T(\frac{a}{2})(y_n)$, and $|T(\frac{a}{2})(y)| > \varepsilon$. Indeed, by definition we have

$$(15) \quad |T(af_n)(y_n)| \geq 2a - \varepsilon - |T(-af_n)(y_n)| \geq a - \varepsilon$$

and by $\|\frac{a}{2} - af_n\| = \frac{a}{2}$ we get

$$(16) \quad T(af_n)(y_n) - \frac{a}{2} \leq T(\frac{a}{2})(y_n) \leq T(af_n)(y_n) + \frac{a}{2}.$$

Hence

$$T(\frac{a}{2})(y_n) \geq a - \varepsilon - \frac{a}{2} > \varepsilon, \text{ if } T(af_n)(y_n) \geq 0.$$

Similarly,

$$T(\frac{a}{2})(y_n) \leq -a + \varepsilon + \frac{a}{2} < -\varepsilon, \text{ if } T(af_n)(y_n) \leq 0.$$

Thus

$$\lim_{n \rightarrow \infty} \text{sign } T(af_n)(y_n) = \text{sign } T(\frac{a}{2})(y).$$

□

Lemma 2.7 ([9]). *Let $f_1, f_2 \in B_{1-\frac{a}{2}}(C(X))$, $x_0 \in X$ and*

$$\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|$$

and $d > 0$ be such that $|f_i(x) - f_i(x_0)| \leq a$, $i = 1, 2$, $x \in B(x_0, d)$. For each n , let

$$p_n(x) = \begin{cases} 1 - \frac{nd_1(x_0, x)}{d}, & x \in B(x_0, \frac{d}{n}) \\ \min_{1,2} \{1 - f_i(x_0) + f_i(x), 1 - a\}, & \text{otherwise.} \end{cases}$$

$$q_n(x) = \begin{cases} -1 + \frac{nad_1(x_0, x)}{d}, & x \in B(x_0, \frac{d}{n}) \\ \max_{1,2} \{-1 - f_i(x_0) + f_i(x), -1 + a\}, & \text{otherwise.} \end{cases}$$

$$r_n(x) = \begin{cases} 1 - \frac{nd_1(x_0, x)}{d}, & x \in B(x_0, \frac{d}{n}) \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \|f_i - p_n\| &\rightarrow 1 - f_i(x_0) \quad (n \rightarrow \infty), \\ \|f_i - q_n\| &\rightarrow 1 + f_i(x_0) \quad (n \rightarrow \infty), \\ \|p_n - ar_n\| &= 1 - a, \\ \|q_n + ar_n\| &= 1 - a. \end{aligned}$$

Lemma 2.8. *Given $x_0 \in X$, let $f_1, f_2 \in B_{1-\frac{a}{2}}(C(X))$, and*

$$\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|.$$

Then there exists a signal function $s: \bigcup_{x \in X} A_x \rightarrow \{-1, 1\}$ and $y_0 \in \varphi^{-1}(x_0)$ such that $|T(f_i)(y_0) - s(y_0)f_i(x_0)| \leq \varepsilon$, $i = 1, 2$.

Proof. Let $K = \bigcup_{x \in X} A_x$, $s(y) = \text{sign } T(\frac{a}{2})(y)$ on K and let $x_0 \in X$, $f_1, f_2 \in B_{1-\frac{a}{2}}(C(X))$ such that $\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|$ and p_n, q_n, r_n are the functions in Lemma 2.7. Clearly, $p_n, q_n \in B_1(C(X))$ and $\|p_n - q_n\| = 2$.

Because T is the ε -isometry, there exist $y_n \in Y$ for every n such that

$$(17) \quad 2 - \varepsilon \leq |T(p_n)(y_n) - T(q_n)(y_n)| \leq 2.$$

The sequence $\{y_n\}$ contains a convergent subsequence, say $y_n \rightarrow y_0$. We shall now prove that

$$y_0 \in \varphi^{-1}(x_0) = A_{x_0}.$$

Since r_n is a tentfunction at x_0 , $\frac{d}{n} \rightarrow 0$ and $y_n \rightarrow y_0$ we have $y_0 \in \varphi^{-1}(x_0) = A_{x_0}$ if we can prove that $-|T(ar_n)(y_n) - T(-ar_n)(y_n)| \leq -2a + \varepsilon$.

Assume that $T(p_n)(y_n) \geq T(q_n)(y_n)$. By (17) we obtain

$$2 - \varepsilon \leq T(p_n)(y_n) - T(q_n)(y_n),$$

therefore

$$\begin{aligned} -|T(ar_n)(y_n) - T(-ar_n)(y_n)| &\leq T(-ar_n)(y_n) - T(ar_n)(y_n) \\ &\leq T(-ar_n)(y_n) - T(q_n)(y_n) + T(p_n)(y_n) \\ &\quad - T(ar_n)(y_n) + T(q_n)(y_n) - T(p_n)(y_n) \\ &\leq 1 - a + 1 - a + \varepsilon - 2 = -2a + \varepsilon. \end{aligned}$$

Thus $y_0 \in \varphi^{-1}(x_0) = A_{x_0}$.

The case $T(p_n)(y_n) \leq T(q_n)(y_n)$ is proved similarly. We shall now prove that

$$|T(f_i)(y_0) - s(y_0)f_i(x_0)| \leq \varepsilon, \quad i = 1, 2.$$

$|T(p_n)(y_n)| \leq 1$ and $|T(q_n)(y_n)| \leq 1$ imply

$$(18) \quad \begin{cases} 1 - \varepsilon \leq T(p_n)(y_n) \leq 1, \\ -1 \leq T(q_n)(y_n) \leq \varepsilon - 1. \end{cases}$$

or

$$(19) \quad \begin{cases} -1 \leq T(p_n)(y_n) \leq \varepsilon - 1, \\ 1 - \varepsilon \leq T(q_n)(y_n) \leq 1. \end{cases}$$

One can easily check that $\text{sign } T(p_n)(y_n) = \text{sign } T(ar_n)(y_n)$. In fact, since

$$\|p_n - ar_n\| = 1 - a,$$

then

$$|T(p_n)(y_n) - T(ar_n)(y_n)| \leq 1 - a.$$

From (18) and (19) we see

if $T(p_n)(y_n) \geq 1 - \varepsilon$,

$$(20) \quad T(ar_n)(y_n) \geq a - 2\varepsilon > 0.$$

if $T(p_n)(y_n) \leq -1 + \varepsilon$,

$$(21) \quad T(ar_n)(y_n) \leq -a + 2\varepsilon < 0.$$

By Lemma 2.6, $s(y_0) = \lim_{n \rightarrow \infty} \text{sign } T(p_n)(y_n)$, so for n large enough we have

$$(22) \quad s(y_0) = \text{sign } T(p_n)(y_n).$$

Hence for n large enough those inequalities can be rewritten in the form

$$\begin{aligned} 1 &\geq s(y_0)T(p_n)(y_n) \geq 1 - \varepsilon \\ -1 + \varepsilon &\geq s(y_n)T(q_n)(y_n) \geq -1. \end{aligned}$$

From Lemma 2.8 we obtain

$$\begin{aligned} -\varepsilon(n, f_i) + T(p_n)(y_n) - \varepsilon - (1 - f_i(x_0)) &\leq T(f_i)(y_n) \\ &\leq 1 - f_i(x_0) + T(p_n)(y_n) + \varepsilon(n, f_i); \\ -\varepsilon(n, f_i) + T(q_n)(y_n) - \varepsilon - (1 + f_i(x_0)) &\leq T(f_i)(y_n) \\ &\leq 1 + f_i(x_0) + T(q_n)(y_n) + \varepsilon(n, f_i), \end{aligned}$$

where $\varepsilon(n, f_i) \rightarrow 0$ when $n \rightarrow \infty$. Hence for n large enough we have

$$-\varepsilon(n, f_i) - \varepsilon + s(y_0)f_i(x_0) \leq T(f_i)(y_n) \leq \varepsilon + s(y_0)f_i(x_0) + \varepsilon(n, f_i).$$

Letting $n \rightarrow \infty$ we obtain

$$|T(f_i)(y_0) - s(y_0)f_i(x_0)| \leq \varepsilon.$$

The proof is complete. $\square$

Before the proof of the Theorem 2.1, we recall the famous *Michael Selected Theorem* [7]. Suppose that Ω is a paracompact and X is a Banach space, if F is a lower-semi-continuous multi-valued function on Ω , and $f(t)$ ($\forall t \in \Omega$) is a closed convex set of X , then there exists a continuous function f satisfies $f(t) \in F(t)$ ($t \in \Omega$).

The proof of Theorem 2.1. Let φ and s be as above. Since

$$s: K = \bigcup_{x \in X} A_x \rightarrow \{-1, 1\}$$

and K is closed we can find, by *Urysohn's Lemma*, a continuous function

$$\bar{s}: Y \rightarrow [-1, 1]$$

with $\bar{s}|_K = s$.

Now, let $M_1(X) = B_1(C(X))^*$ be the unit ball of the Radon measure space on X endowed with the *weak**-topology. Define a set valued map on Y , $\Psi: Y \rightarrow 2^{M_1(X)}$ by

$$\Psi(y) = \begin{cases} s(y)\delta_{\varphi(y)}, & y \in K, \\ \{\bar{s}(y)\mu, \mu \text{ is the probability measure of } M_1(X)\}, & y \in Y \setminus K. \end{cases}$$

Clearly $\Psi(y)$ is a closed and convex subset of $M_1(X)$ for all $y \in Y$. Furthermore, we can check that the set is the *w**- lower-semi-continuous.

Assume that $y_n \rightarrow y$ when $n \rightarrow \infty$ and $\nu \in \Psi(y)$. Thus

$$\nu = \begin{cases} s(y)\delta_{\varphi(y)}, & y \in K, \\ \bar{s}(y)\mu, \mu \text{ is some probability measure of } M_1(X), & y \in Y \setminus K. \end{cases}$$

Let

$$\nu_n = \begin{cases} s(y_n)\delta_{\varphi(y_n)}, & y_n \in K, \\ \bar{s}(y_n)\mu', & y_n \in Y \setminus K. \end{cases}$$

Where

$$\mu' = \begin{cases} \delta_{\varphi(y)}, & y \in K, \\ \mu, & y \in Y \setminus K \end{cases}$$

is the probability measure of $M_1(X)$, hence $\nu_n \in \varphi_n(y_n)$.

We shall now prove that $\nu_n \xrightarrow{w^*} \nu$ when $n \rightarrow \infty$.

(1) If $y \in K$ and there is a subsequence $\{y_n\} \subset K$, φ is continuous implies $\delta_{\varphi(y_n)} \xrightarrow{w^*} \delta_{\varphi(y)}$ by $\nu_n \xrightarrow{w^*} \nu$ when $n \rightarrow \infty$.

(2) If $y \in K$ and there is a subsequence $\{y_n\} \subset Y \setminus K$, then $\nu_n = \bar{s}(y_n)\delta_{\varphi(y_n)} \xrightarrow{w^*} \nu$ when $n \rightarrow \infty$.

(3) If $y \notin K$, since $Y \setminus K$ is an open set, then it is necessary there exists N such that $y_n \in Y \setminus K$ for $n > N$, hence $\nu_n = \bar{s}(y_n)\mu \xrightarrow{w^*} \nu$ when $n \rightarrow \infty$.

We can find, by *Michael Selected Theorem*, a *w**- continuous function

$$\tilde{\Psi}: Y \rightarrow M_1(X),$$

satisfies $\tilde{\Psi}(y) \in \Psi(y)$. Furthermore we have that $\tilde{\Psi}(y) = s(y)\delta_{\varphi(y)}$ for all $y \in K$.

Now, for any $y \in Y, f \in B_{1-\frac{\varepsilon}{2}}(C(X))$ define a map by

$$U(f)(y) = \sup\{\inf\{\tilde{\Psi}(y)(f), T(f)(y) + \varepsilon\}, T(f)(y) - \varepsilon\}.$$

Clearly $|T(f)(y) - \tilde{\Psi}(y)(f)| \leq \varepsilon$ if and only if $U(f)(y) = \tilde{\Psi}(y)(f)$.

Since $\tilde{\Psi}(y)$ is *w**- continuous, we have $U(f)(y)$ is continuous on Y and hence $U(f) \in C(Y)$. We now prove that U is an isometry and to do this we first show that

$$(23) \quad |U(f_1)(y) - U(f_2)(y)| \leq \|f_1 - f_2\|, \quad \forall y \in Y.$$

(1) If $U(f_i)(y) = \tilde{\Psi}(y)(f_i)$, (23) is true.

(2) If $U(f_i)(y) = T(f_i(y)) \mp \varepsilon$, let $U(f_1)(y) = T(f_1(y)) - \varepsilon$, and $U(f_2)(y) = T(f_2(y)) + \varepsilon$, then by definition of $U(f)(y)$

$$\begin{cases} U(f_1)(y) \geq \tilde{\Psi}(y)(f_1), \\ U(f_2)(y) \leq \tilde{\Psi}(y)(f_2). \end{cases}$$

Hence

$$U(f_2)(y) - U(f_1)(y) \leq \tilde{\Psi}(y)(f_2 - f_1) \leq \|f_1 - f_2\|,$$

$$U(f_1)(y) - U(f_2)(y) \leq \|f_1 - f_2\|.$$

(3) If $U(f_1)(y) = \tilde{\Psi}(y)(f_1)$ and $U(f_2)(y) = T(f_2(y)) + \varepsilon$, then

$$U(f_1)(y) \leq T(f_1(y)) + \varepsilon,$$

thus

$$U(f_2)(y) - U(f_1)(y) \leq \tilde{\Psi}(y)(f_2 - f_1) \leq \|f_1 - f_2\|,$$

$$U(f_1)(y) - U(f_2)(y) \leq T(f_1)(y) + \varepsilon - T(f_2)(y) - \varepsilon \leq \|f_1 - f_2\|.$$

(4) The case $U(f_1)(y) = \tilde{\Psi}(y)(f_1)$, $U(f_2)(y) = T(f_2)(y) - \varepsilon$ is proved similarly.

Now we shall prove that

$$(24) \quad \|U(f_1) - U(f_2)\| \geq \|f_1 - f_2\|.$$

Given $x_0 \in X$ such that $\|f_1 - f_2\| = |f_1(x_0) - f_2(x_0)|$, then by Lemma 2.8, we can find a point $y_0 \in \varphi^{-1}(x_0) = A_{x_0} \subset K$ such that

$$|T(f_i)(y_0) - s(y_0)f_i(x_0)| \leq \varepsilon$$

and $s(y_0)f_i(x_0) = s(y_0)\delta_{\varphi(y_0)}f_i = \tilde{\Psi}(y_0)(f_i)$. Thus $U(f_i)(y_0) = \tilde{\Psi}(y_0)(f_i)$. Hence

$$\|U(f_1) - U(f_2)\| \geq |s(y_0)f_1(x_0) - s(y_0)f_2(x_0)| = \|f_1 - f_2\|.$$

Furthermore, for any $f \in B_{1-\frac{\varepsilon}{2}}(C(X))$ we have

$$\|T(f) - U(f)\| \leq \varepsilon$$

(1) $U(f)(y) = \tilde{\Psi}(y)f$ is equivalent to

$$|T(f)(y) - U(f)(y)| \leq \varepsilon,$$

(2) If $U(f)(y) = T(f)(y) \pm \varepsilon$, clearly,

$$\|T(f) - U(f)\| \leq \varepsilon$$

and the proof is complete. $\square$

3. THE COUNTEREXAMPLES FOR ε - ISOMETRIC APPROXIMATE PROBLEM.

Theorem 3.1. *Let $M \geq 3$, and any $\varepsilon > 0$. Then there exists an ε -isometry*

$$T: B_1(l_1) \rightarrow B_1(l_1)$$

such that for any isometry U which defines on some subset of l_1 that contains $B_{\frac{\varepsilon}{M}}(l_1)$, it is necessary to have $x \in B_{\frac{3}{M}}(l_1)$ with $\|Tx - Ux\| \geq \frac{2}{M^2}$.

Theorem 3.2. *Let $M \geq 3$, for any $\varepsilon > 0$, then there exists an ε -isometry*

$$T: B_1((L_1(0, 1) \times R)_1) \rightarrow B_1((L_1(0, 1) \times R)_1)$$

such that for any isometry U which defines on some subset of $(L_1(0, 1) \times R)_1$ that contains $B_{\frac{\varepsilon}{M}}((L_1(0, 1) \times R)_1)$, it is necessary to have $x \in B_{\frac{3}{M}}((L_1(0, 1) \times R)_1)$ with $\|Tx - Ux\| \geq \frac{2}{M^2}$ (where $\|(f, r)\| = \|f\|_{L_1} + |r|$) is the norm of $(L_1(0, 1) \times R)_1$.

Lemma 3.3 ([10]). Let $n \in \mathbf{N}$, $\varepsilon = \frac{1}{n}$ and $a \in l_1$, $S_a = \{1, 2, \dots, n\} \cap \text{supp}(a)$.

Let

$$T_1(a) = \begin{cases} a, & a_{n+1} < 0, \\ a + \frac{a_{n+1}}{M}(\varepsilon \sum e_i - e_{n+1}), & a_{n+1} > 0. \end{cases}$$

For any $a, b \in l_1$, if $\text{card}(S_a), \text{card}(S_b) \leq \frac{M}{2}$ and $a_i, b_i \geq 0$, ($1 \leq i \leq n$). Then

1) if $a_{n+1}, b_{n+1} \leq 0$, then

$$\|T_1(a) - T_1(b)\| = \|a - b\|,$$

2) if $a_{n+1}, b_{n+1} \geq 0$, then

$$\|a - b\| \geq \|T_1(a) - T_1(b)\| \geq \|a - b\| - 2\varepsilon(\text{card}(S_b)) \frac{(a_{n+1} - b_{n+1})}{M},$$

3) if $a_{n+1} \geq 0 \geq b_{n+1}$, then

$$\|a - b\| \geq \|T_1(a) - T_1(b)\| \geq \|a - b\| - 2\varepsilon \sum_{S_b} \frac{a_{n+1}}{M}.$$

Furthermore

$$\|a - b\| \geq \|T_1(a) - T_1(b)\| \geq \|a - b\|(1 - \varepsilon).$$

Lemma 3.4 ([10]). Let $n \in \mathbf{N}$, $\varepsilon = \frac{1}{n}$, $a \in l_1$, and $S_a = \{1, 2, \dots, n\} \cap \text{supp}(a)$.

Let $T_2(a) = \sum_{i=1}^{\infty} T_2(a_i e_i)$, where

$$(25) \quad T_2(a_i e_i) = \begin{cases} a_i e_{n+1+i}, & i \leq n \text{ and } a_i < \frac{2}{M}, \\ (a_i - \frac{2}{M})e_i + (\frac{2}{M})e_{n+1+i}, & i \leq n \text{ and } a_i \geq \frac{2}{M}, \\ a_{n+1} e_{n+1}, & i = n+1, \\ a_i e_{n+1}, & i > n+1. \end{cases}$$

Then T_2 is an isometry and if $a \in B_1(l_1)$, then $(T_2(a))_i \geq 0$, $1 \leq i \leq n$ and $\text{card}(S_{T_2(a)}) \leq \frac{M}{2}$.

Lemma 3.5 ([10]). Let T_1, T_2 satisfy the conditions of the Lemma 3.3 and Lemma 3.4 and $T = T_1 \circ T_2$. Then for any isometry U which defines on some subset of l_1 that contains $B_{\frac{6}{M}}(l_1)$, it's necessary to have $x \in B_{\frac{3}{M}}(l_1)$, with $\|Tx - Ux\| \geq \frac{2}{M^2}$.

Proof of Theorem 3.1. We should only show that T is an ε -isometry on $B_1(l_1)$. By Lemma 3.3 T_2 is an isometry, and if $a \in B_1(l_1)$, then $(T_2(a))_i \geq 0$, and $\text{card}(S_{T_2(a)}) \leq \frac{M}{2}$.

if $\text{card}(S_a), \text{card}(S_b) \leq \frac{M}{2}$, and $a_i, b_i \geq 0$, then

$$(26) \quad \|a - b\| \geq \|T_1(a) - T_1(b)\| \geq \|a - b\| - \varepsilon.$$

Directly by Lemma 3.4 we get

(1) If $a_{n+1}, b_{n+1} \leq 0$,

$$\|T_1(a) - T_1(b)\| = \|a - b\|,$$

(2) If $a_{n+1} \geq b_{n+1} \geq 0$,

$$\|a - b\| \geq \|T_1(a) - T_1(b)\| \geq \|a - b\| - 2\varepsilon(\text{card}(S_b)) \frac{(a_{n+1} - b_{n+1})}{M}$$

Since $\text{card}(S_b) \leq \frac{M}{2}$, clearly, $a_{n+1} - b_{n+1} \leq a_{n+1} < 1$.

(3) If $a_{n+1} \geq 0 \geq b_{n+1}$,

$$\|a - b\| \geq \|T_1(a) - T_1(b)\| \geq \|a - b\| - 2 \sum_{S_b} \varepsilon \frac{a_{n+1}}{M} \geq \|a - b\| - \varepsilon, \quad (a_{n+1} \leq 1).$$

Thus T is an ε -isometry. By Lemma 3.5, for any isometry U which defines on subset of l_1 that contains $B_{\frac{6}{M}}(l_1)$. It is necessary to have $x \in B_{\frac{3}{M}}(l_1)$ with $\|Tx - Ux\| \geq \frac{2}{M^2}$ $\square$

Remark 3.6. The proof for Theorem 3.2 is gotten by revising Lövblom's [10] method.

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