

$(n - 1)$ -DIMENSIONAL GENERALIZED NULL SCROLLS IN R_1^n

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ABSTRACT. In this paper, we obtained relationships between the principal curvatures of an $(n - 1)$ -dimensional generalized null scroll M which is a ruled hypersurface in R_1^n . We calculated the normal curvature of M and a characterized the curvature lines.

1. PRELIMINARIES

Let M be an m -dimensional Lorentzian submanifold of R_1^n . Let $\bar{\nabla}$ and ∇ denote the Levi-Civita connections of R_1^n and M , respectively. For any vector fields X, Y tangent to M we have the Gauss formula

$$(1.1) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where h denotes the second fundamental form of M in R_1^n . Our second fundamental equation is the Weingarten formula

$$(1.2) \quad \bar{\nabla}_X \xi = -A_\xi X + D_X \xi$$

where ξ is a normal vector field to M , A_ξ is the Weingarten map with respect to ξ and D is the normal connection of M [4]. It is also well-known that h and A are related by

$$(1.3) \quad \langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$$

Let $\{\xi_1, \xi_2, \dots, \xi_{n-m}\}$ be a local orthonormal frame field for $\chi^\perp(M)$. Then the mean curvature vector field H of M in R_1^n is given by [4]

$$(1.4) \quad H = \sum_{j=1}^{n-m} \frac{\text{trace } A_{\xi_j}}{m} \xi_j.$$

Let ξ be a unit normal vector field to M . The Lipschitz-Killing curvature in the direction ξ at a point $p \in M$ is defined by

$$(1.5) \quad G(p, \xi) = \det A_\xi(p).$$

while the Gauss curvature at p is

$$(1.6) \quad G(p) := \sum_{j=1}^{n-m} G(p, \xi_j).$$

If $G(p) = 0$ for all $p \in M$, we say that M is developable [5].

Let M be a n -dimensional Lorentzian manifold and $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_p(M)$, $p \in M$. The scalar curvature of M is defined by

$$(1.7) \quad \mathbf{r} = \sum_{i=1}^n \varepsilon_i \operatorname{Ric}(e_i, e_i)$$

where

$$\varepsilon_i = \langle e_i, e_i \rangle, \quad \varepsilon_i = \begin{cases} -1, & \text{if } e_i \text{ is timelike} \\ 1, & \text{if } e_i \text{ is spacelike} \end{cases}$$

and Ric is the Ricci curvature tensor field of M [4].

Now, suppose that M is a hypersurfaces in R_1^n and let A be the shape tensor of M .

The normal curvature of M along a unit tangent direction X_p in $T_p M$ is defined by

$$(1.8) \quad k_n(X_p) = \langle A_p(X_p), X_p \rangle$$

Let α be a null curve on the hypersurfaces M . According to [4], if the equality

$$(1.9) \quad \langle A(\alpha(t)), \alpha(t) \rangle = 0$$

is satisfied, then α is said to be an asymptotic curve on M . The null curve α is called a line of curvature on M if

$$(1.10) \quad A\alpha' = k.\alpha' \quad (k \in \mathbf{R}^*).$$

2. $(n-1)$ -DIMENSIONAL GENERALIZED NULL SCROLLS IN R_1^n AND THEIR CURVATURES

We recall the notion of a generalized null scroll in R_1^n [1]. Let M be an $(n-1)$ -dimensional generalized null scroll in R_1^n and suppose that the base null curve α is an pseudo-orthogonal trajectory of the generating space of M . Then we may parametrize M as

$$(2.1) \quad \varphi(t, u_0, u_1, \dots, u_{n-3}) = \alpha(t) + u_0 Y(t) + \sum_{i=1}^{n-3} u_i Z_i(t)$$

where $\{Y(t), Z_1(t), \dots, Z_{n-3}(t)\}$ is the null basis of the generating space and

$$\{X(t), Y(t), Z_1(t), \dots, Z_{n-3}(t)\}$$

with $X = \varphi_*(\frac{\partial}{\partial t})$ is pseudo-orthonormal basis of $T_{\varphi(t)}M$. We recall that a basis $\{X, Y, Z_1, \dots, Z_{n-2}\}$ of R_1^n is said to be pseudo-orthonormal if the following conditions are fulfilled [3]:

$$\begin{aligned} \langle X, X \rangle &= \langle Y, Y \rangle = 0; & \langle X, Y \rangle &= -1 \\ \langle X, Z_i \rangle &= \langle Y, Z_i \rangle = 0; & \text{for } 1 \leq i \leq n-2 \\ \langle Z_i, Z_j \rangle &= \delta_{ij}, & \text{for } 1 \leq i \leq n-2. \end{aligned}$$

The matrix of the shape operator $A_{\varphi(t)}$ with respect to this basis is of the form

$$(2.2) \quad M(A_{\varphi(t)}) = - \begin{bmatrix} b_{10} & b_{00} & c_{01} & c_{02} & \cdots & c_{0(n-3)} \\ 0 & b_{10} & 0 & 0 & \cdots & 0 \\ 0 & b_{11} & 0 & 0 & \cdots & 0 \\ 0 & b_{12} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{1(n-3)} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, the characteristic equation of A is

$$\det(A - \lambda I_{n-1}) = \det \begin{bmatrix} -b_{10} - \lambda & -b_{00} & -c_{01} & -c_{02} & \cdots & -c_{0(n-3)} \\ 0 & -b_{10} - \lambda & 0 & 0 & \cdots & 0 \\ 0 & -b_{11} & -\lambda & 0 & \cdots & 0 \\ 0 & -b_{12} & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -b_{1(n-3)} & 0 & 0 & \cdots & -\lambda \end{bmatrix} = 0,$$

which leads to the relation

$$(2.3) \quad (b_{10} + \lambda)^2 (-1)^{n-3} \lambda^{n-3} = 0.$$

Since from equation (2.2) $\text{rank } A = (n - 3)$, the eigenvalues of A are

$$(2.4) \quad \lambda_3 = \lambda_4 = \cdots = \lambda_{n-1} = 0,$$

hence we obtain

$$(2.5) \quad \lambda_1 = \lambda_2 = -b_{10}$$

and

$$\lambda_1 + \lambda_2 = 2\lambda_1 = 2\lambda_2 = -2b_{10},$$

therefore

$$(2.6) \quad \text{trace } A = -2b_{10}$$

and

$$(2.7) \quad H = -\frac{2}{n-1}b_{10}.$$

If M is a minimal hypersurface, then $H = 0$ and we get $b_{10} = 0$, i.e., $\lambda_1 = \lambda_2 = 0$. Thus we have the following

Corollary 2.1. *If an $(n - 1)$ -dimensional generalized null scroll M is minimal, then the principal curvatures of M vanish at any point.*

From the equation (2.5) we infer

Corollary 2.2. *For an $(n - 1)$ -dimensional generalized null scroll, the principal curvatures are equal.*

Theorem 2.1. *Let M be an $(n - 1)$ -dimensional generalized null scroll. Then the scalar curvature of M is*

$$r = -2 \left\{ \sum_{i=1}^{n-3} (b_{1i})^2 + \lambda^2 \right\},$$

where $\lambda_1 = \lambda_2 = \lambda$.

Proof. If we take $j = 1$ in formula (4.16) of [1] we obtain

$$r = -2 \sum_{i=1}^{n-3} (b_{1i})^2 - 2(b_{10})^2.$$

By equation (2.5), we have

$$r = -2 \sum_{i=1}^{n-3} (b_{1i})^2 - 2\lambda_1^2$$

or

$$r = -2 \sum_{i=1}^{n-3} (b_{1i})^2 - 2\lambda_2^2.$$

Also, since $\lambda_1 = \lambda_2$, we find

$$r = -2 \left\{ \sum_{i=1}^{n-3} (b_{1i})^2 + \lambda^2 \right\},$$

as was to be shown. $\square$

For $n \geq 4$, $\det A = 0$ by (2.2), so we have the following

Corollary 2.3. *The Gauss curvature of an $(n-1)$ -dimensional generalized null scroll is identically zero if $n \geq 4$.*

Corollary 2.4. *The Lipschitz–Killing curvature of an $(n-1)$ -dimensional generalized null scroll in the normal direction is equal to the Gauss curvature.*

Proof. If we take $j = 1$ in formula (4.11) of [1], the proof is clear. $\square$

Now let $V_p \in T_p M$ and suppose that $V_p = (v, v_0, v_1, \dots, v_{n-3})$. From equation (2.2) we obtain

$$\begin{aligned} A(V_p) &= - \begin{bmatrix} b_{10} & b_{00} & c_{01} & c_{02} & \cdots & c_{0(n-3)} \\ 0 & b_{10} & 0 & 0 & \cdots & 0 \\ 0 & b_{11} & 0 & 0 & \cdots & 0 \\ 0 & b_{12} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{1(n-3)} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v \\ v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{n-3} \end{bmatrix} \\ &= (-b_{10}v - b_{00}v_0 - \sum_{i=1}^{n-3} c_{0i}v_i, -b_{10}v_0, -b_{11}v_0, \dots, -b_{1(n-3)}v_0), \end{aligned}$$

while by equation (1.8) we get

$$k_n(V_p) = \left(-\frac{n-1}{2}Hv + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i\right)v - \left(\sum_{i=0}^{n-3} b_{1i}v_i\right)v_0.$$

These relations lead to the following

Corollary 2.5. *Let M be an $(n-1)$ -dimensional generalized null scroll. Then a direction V_p , whose first and second components are zero, is an asymptotic direction in M .*

Theorem 2.2. *Let M be an $(n-1)$ -dimensional generalized null scroll. Then a direction $V_p = (v, v_0, v_1, \dots, v_{n-3})$ of M is principal curvature direction for M at p if and only if*

$$(2.8) \quad -\frac{b_{1j}v_0}{v_j} + \frac{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}{v} = 0, \quad j = 0, 1, \dots, n-3,$$

where b_{00}, b_{1j}, c_{0i} , are elements of the matrix of A .

Proof. If V_p is principal curvature direction for M at p , then from equation (1.10) we have

$$(-b_{10}v - b_{00}v_0 - \sum_{i=1}^{n-3} c_{0i}v_i, -b_{10}v_0, -b_{11}v_0, \dots, -b_{1(n-3)}v_0) = k(v, v_0, v_1, \dots, v_{n-3}).$$

Thus we obtain

$$(2.9) \quad k = -\frac{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}{v}$$

and

$$(2.10) \quad k = -\frac{b_{1j}v_0}{v_j}, \quad j = 0, 1, \dots, n-3.$$

From equations (2.9) and (2.10), we get

$$(2.11) \quad -\frac{b_{1j}v_0}{v_j} + \frac{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}{v} = 0, \quad j = 0, 1, \dots, n-3.$$

Conversely, let us assume that equation (2.8) is satisfied. Then we have

$$(2.12) \quad v_j = \frac{b_{1j}v_0v}{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}, \quad j = 0, 1, \dots, n-3;$$

and

$$A(V_p) = -\frac{b_{10}v + b_{00}v_0 + \sum_{i=1}^{n-3} c_{0i}v_i}{v} V_p$$

which concludes the proof. $\square$

Corollary 2.6. *A curve β in an $(n-1)$ -dimensional generalized null scroll M is a line of curvature if and only if it satisfies the following system of differential equations:*

$$(2.13) \quad -b_{1j} \frac{d\beta_0}{dt} \frac{d\beta_j}{dt} + \frac{d\beta_j}{dt} (b_{10} \frac{d\beta}{dt} + b_{00} \frac{d\beta_0}{dt} + \sum_{i=1}^{n-3} c_{0i} \frac{d\beta_i}{dt}) = 0, \quad j = 0, 1, \dots, n-3.$$

Proof. This is clear by putting $v = \frac{d\beta}{dt}$, $v_k = \frac{d\beta_k}{dt}$, $k = 0, 1, \dots, n-3$, in Theorem 2.2. $\square$

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