

SOME INEQUALITIES FOR RICCI CURVATURE OF CERTAIN SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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ABSTRACT. In the present paper, we obtain sharp inequalities between the Ricci curvature and the squared mean curvature for slant, semi-slant and bi-slant submanifolds in Sasakian space forms. Also, estimates of the scalar curvature and the k -Ricci curvature respectively, in terms of the squared mean curvature, are proved.

1. PRELIMINARIES

A $(2m + 1)$ -dimensional Riemannian manifold (\tilde{M}, g) is said to be a *Sasakian manifold* if it admits an endomorphism ϕ of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η , satisfying:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \\ (\tilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \quad \tilde{\nabla}_X \xi = \phi X, \end{cases}$$

for any vector fields X, Y on $T\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g .

A plane section π in $T_p\tilde{M}$ is called a ϕ -*section* if it is spanned by X and ϕX , where X is a unit tangent vector orthogonal to ξ . The sectional curvature of a ϕ -section is called a ϕ -*sectional curvature*. A Sasakian manifold with constant ϕ -sectional curvature c is said to be a *Sasakian space form* and is denoted by $\tilde{M}(c)$.

The curvature tensor of $\tilde{M}(c)$ of a Sasakian space form $\tilde{M}(c)$ is given by [1]

$$\begin{aligned} \tilde{R}(X, Y)Z &= \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \\ (1.1) \quad &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned}$$

for any tangent vector fields X, Y, Z on $\tilde{M}(c)$.

As examples of Sasakian space forms we mention \mathbb{R}^{2m+1} and S^{2m+1} , with standard Sasakian structures (see [1]).

In [9], A. Lotta has introduced the following notion of slant immersion in almost contact metric manifolds.

Definition. We call a differentiable distribution \mathcal{D} on M a *slant distribution* if for each $x \in M$ and each nonzero vector $X \in \mathcal{D}_x$, the angle $\theta_{\mathcal{D}}(X)$ between ϕX and the vector subspace \mathcal{D}_x is constant, which is independent of the choice of $x \in M$

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and $X \in \mathcal{D}_x$. In this case, the constant angle $\theta_{\mathcal{D}}$ is called the *slant angle* of the distribution \mathcal{D} .

Definition. A submanifold M tangent to ξ is said to be *slant* if for any $x \in M$ and any $X \in T_x M$, linearly independent of ξ , the angle between ϕX and $T_x M$ is a constant $\theta \in [0, \frac{\pi}{2}]$, called the *slant angle* of M in \tilde{M} .

Examples of slant submanifolds. (see [2]).

Example 1. For any constant k ,

$$x(u, v, t) = 2(e^{ku} \cos u \cos v, e^{ku} \sin u \cos v, e^{ku} \cos u \sin v, e^{ku} \sin u \sin v, t)$$

defines a slant submanifold of dimension 3 with slant angle $\theta = \arccos \frac{|k|}{\sqrt{1+k^2}}$, scalar curvature $\tau = \frac{-k^2}{3(1+k^2)}$ and non-constant mean curvature given by $\|H\| = \frac{2e^{-ku}}{3\sqrt{1+k^2}}$. Hence, the submanifold is not minimal.

Example 2. For any constant k ,

$$x(u, v, t) = 2(u, k \cos v, v, k \sin v, t)$$

defines a slant submanifold M with slant angle $\theta = \arccos \frac{1}{\sqrt{1+k^2}}$, scalar curvature $\tau = \frac{-1}{3(1+k^2)}$, constant mean curvature given by $\|H\| = \frac{|k|}{3(1+k^2)}$. Moreover, the following statements are equivalent:

- (a) $k = 0$;
- (b) M is invariant;
- (c) M is minimal;
- (d) M has parallel mean curvature vector.

Invariant and *anti-invariant immersions* are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a *proper slant immersion*.

Definition. We say that a submanifold M tangent to ξ is a *bi-slant* submanifold of \tilde{M} if there exist two orthogonal distributions \mathcal{D}_1 and \mathcal{D}_2 on M such that :

- i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.
- ii) For any $i = 1, 2$, \mathcal{D}_i is slant distribution with slant angle θ_i .

Let $2d_1 = \dim \mathcal{D}_1$ and $2d_2 = \dim \mathcal{D}_2$.

Remark. If either d_1 or d_2 vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds (and, therefore, invariant and anti-invariant submanifolds) are particular cases of bi-slant submanifolds.

Examples of bi-slant submanifolds. (see [2], [3])

Example 1. For any $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$,

$$x(u, v, w, s, t) = 2(u, 0, w, 0, v \cos \theta_1, v \sin \theta_1, s \cos \theta_2, s \sin \theta_2, t)$$

defines a five-dimensional bi-slant submanifold M , with slant angles θ_1 and θ_2 , is \mathbf{R}^9 with its usual Sasakian structure (ϕ_0, ξ, η, g) .

Furthermore, it is easy to see that

$$\begin{aligned} e_1 &= 2\left(\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}\right), & e_2 &= \cos \theta_1 \left(2\frac{\partial}{\partial y^1}\right) + \sin \theta_1 \left(2\frac{\partial}{\partial y^2}\right), \\ e_3 &= 2\left(\frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}\right), & e_4 &= \cos \theta_2 \left(2\frac{\partial}{\partial y^3}\right) + \sin \theta_2 \left(2\frac{\partial}{\partial y^4}\right), \\ e_5 &= 2\frac{\partial}{\partial z} = \xi, \end{aligned}$$

form a local orthonormal frame of TM . We define the distributions $\mathcal{D}_1 = \langle e_1, e_2 \rangle$ and $\mathcal{D}_2 = \langle e_3, e_4 \rangle$.

Then, it is clear that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \langle \xi \rangle$ and it can be easily proved that \mathcal{D}_i is a slant distribution with slant angle θ_i for any $i = 1, 2$. In particular, if we consider $\theta_1 = \theta_2 = \theta$ in the above, it results that M is a θ -slant submanifold.

Example 2. For any $\theta_1 \in [0, \frac{\pi}{2}]$, we chose $\theta_2 \in (0, \frac{\pi}{2}]$, such that $\cos \theta_2 = \frac{\cos \theta_1}{\sqrt{2}}$. Then

$$x(u, v, w, s, t) = 2(u, 0, w, 0, v \cos \theta_1, v \sin \theta_1, s \cos \theta_2, s \sin \theta_2, t)$$

defines a five-dimensional bi-slant submanifold M in $(\mathbf{R}^9, \phi_0, \xi, \eta, g)$, with both slant angles equal to θ_2 , but it is not slant submanifold. In fact we can chose a local orthonormal frame $\{e_1, \dots, e_5\}$ of TM such that

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{2}} \left\{ 2\left(\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}\right) + 2\left(\frac{\partial}{\partial x^4} + y^4 \frac{\partial}{\partial z}\right) \right\}, & e_2 &= \cos \theta_1 \left(2\frac{\partial}{\partial y^1}\right) + \sin \theta_1 \left(2\frac{\partial}{\partial y^2}\right), \\ e_3 &= 2\left(\frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}\right), & e_4 &= \cos \theta_2 \left(2\frac{\partial}{\partial y^3}\right) + \sin \theta_2 \left(2\frac{\partial}{\partial y^4}\right), \\ e_5 &= 2\frac{\partial}{\partial z} = \xi. \end{aligned}$$

Now we define the distributions $\mathcal{D}_1 = \langle e_1, e_2 \rangle$ and $\mathcal{D}_2 = \langle e_3, e_4 \rangle$. It is easy to see that both \mathcal{D}_1 and \mathcal{D}_2 are slant distribution with the same slant angle θ_2 . Nevertheless, we can obtain that M is not slant since $\theta_2 \neq 0$.

Definition. We say that M tangent to ξ is a *semi-slant* submanifold of \tilde{M} if there exist two orthogonal distributions \mathcal{D}_1 and \mathcal{D}_2 on M such that :

- i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.
- ii) The distribution \mathcal{D}_1 is an invariant distribution, i.e., $\phi(\mathcal{D}_1) = \mathcal{D}_1$.
- ii) The distribution \mathcal{D}_2 is slant with angle $\theta \neq 0$.

Let $2d_1 = \dim \mathcal{D}_1$ and $2d_2 = \dim \mathcal{D}_2$.

In [3], the invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that, in fact, semi-slant submanifolds are particular cases of bi-slant submanifolds. Moreover, it is clear that, if $\theta = \frac{\pi}{2}$, then the semi-slant submanifold is a semi-invariant submanifold.

- (a) If $d_2 = 0$, then M is an invariant submanifold.
- (b) If $d_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
- (c) If $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then M is a proper slant submanifold, with slant angle θ .

We say that a semi-slant submanifold is *proper* if $d_1 d_2 \neq 0$ and $\theta \neq \frac{\pi}{2}$.

Examples of semi-slant submanifolds. (see [3])

Example 1. Let \mathbf{R}^6 be the Euclidian space of dimension 6, with the standard metric and the almost complex structure given by $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i}$, for any $i = 1, 2, 3$, where (x^i, y^i) denote the Cartesian coordinates.

Let $\mathbf{R}^5 \hookrightarrow \mathbf{R}^6$ be the usual immersion. Then, $C = \frac{\partial}{\partial y^3}$ is the unit normal to \mathbf{R}^5 and so, $\xi = -JC = \frac{\partial}{\partial x^3}$.

Now, for any $\theta \neq 0$, we can consider the immersions:

$$\begin{aligned} \varphi_1 : \mathbf{R}^4 &\longrightarrow \mathbf{R}^6 : (u, v, t, s) \longmapsto (u \cos \theta, u \sin \theta, t, v, 0, s), \\ \varphi_2 : \mathbf{R}^3 &\longrightarrow \mathbf{R}^5 : (u, v, t) \longmapsto (u \cos \theta, u \sin \theta, t, v, 0). \end{aligned}$$

We can directly prove that φ_1 is a semi-slant immersion, with complex distribution $\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x^3}, \frac{\partial}{\partial y^3} \right\rangle$ and slant distribution, with angle θ ,

$$\mathcal{D}_2 = \left\langle \cos \theta \frac{\partial}{\partial x^1} + \sin \theta \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1} \right\rangle.$$

On the other hand, φ_2 is a θ -slant immersion, where \mathbf{R}^5 has the almost contact metric structure induced by the described almost Hermitian structure on \mathbf{R}^6 .

For the other properties and examples of slant, bi-slant and semi-slant submanifolds in Sasakian manifolds, we refer to [2], [3].

Let M be an n -dimensional submanifold of a Riemannian manifold \tilde{M} . We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M, p \in M$, and ∇ the Riemannian connection of M . Also, let h be the second fundamental form and R the Riemann curvature tensor of M .

Then the equation of Gauss is given by

$$(1.2) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) + \\ &+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned}$$

for any vectors X, Y, Z, W tangent to M .

Let $p \in M$ and $\{e_1, \dots, e_n\}$ an orthonormal basis of the tangent space $T_p M$. We denote by H the mean curvature vector, that is

$$(1.3) \quad H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i).$$

Also, we set

$$(1.4) \quad h_{ij}^r = g(h(e_i, e_j), e_r)$$

and

$$(1.5) \quad \|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).$$

For any tangent vector field X to M , we put $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX , respectively. We denote by

$$(1.6) \quad \|P\|^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j).$$

Suppose L is a k -plane section of $T_p M$ and X a unit vector in L . We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of L such that $e_1 = X$.

Define the *Ricci curvature* Ric_L of L at X by

$$(1.7) \quad \text{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k},$$

where K_{ij} denotes the *sectional curvature of the 2-plane section spanned by e_i, e_j* . We simply called such a curvature a k -Ricci curvature.

The *scalar curvature* τ of the k -plane section L is given by

$$(1.8) \quad \tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}.$$

For each integer k , $2 \leq k \leq n$, the Riemannian invariant Θ_k on an n -dimensional Riemannian manifold M is defined by

$$(1.9) \quad \Theta_k(p) = \frac{1}{k-1} \inf_{L, X} \text{Ric}_L(X), \quad p \in M,$$

where L runs over all k -plane sections in $T_p M$ and X runs over all unit vectors in L .

Recall that for a submanifold M in a Riemannian manifold, the *relative null space of M at a point $p \in M$* is defined by

$$(1.10) \quad \mathcal{N}_p = \{X \in T_p M \mid h(X, Y) = 0, \text{ for all } Y \in T_p M\}.$$

2. RICCI CURVATURE AND SQUARED MEAN CURVATURE

Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [6]).

We prove similar inequalities for slant, bi-slant and semi-slant submanifolds in a Sasakian space form.

We consider submanifolds M tangent to the Reeb vector field ξ .

Theorem 2.1. *Let M be an $(n = 2k + 1)$ -dimensional θ -slant submanifold tangent to ξ in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$. Then:*

(i) *For each unit vector $X \in T_p M$ orthogonal to ξ , we have*

$$(2.1) \quad \text{Ric}(X) \leq \frac{1}{4} \{(n-1)(c+3) + \frac{1}{2}(3 \cos^2 \theta - 2)(c-1) + n^2 \|H\|^2\}.$$

(ii) *If $H(p) = 0$, then a unit tangent vector $X \in T_p M$ orthogonal to ξ satisfies the equality case of (2.1) if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of (2.1) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Proof. Let $X \in T_p M$ be a unit tangent vector X at p , orthogonal to ξ . We choose an orthonormal basis $e_1, \dots, e_n = \xi, e_{n+1}, \dots, e_{2m+1}$ such that e_1, \dots, e_n are tangent to M at p , with $e_1 = X$.

Then, from the equation of Gauss, we have

$$(2.2) \quad n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1) \frac{c+3}{4} - [3(n-1) \cos^2 \theta - 2n + 2] \frac{c-1}{4}.$$

From (2.2), we get

$$\begin{aligned}
 (2.3) \quad n^2 \|H\|^2 &= 2\tau + \sum_{r=n+1}^{2m+1} [(h_{11}^r)^2 + (h_{22}^r + \dots + h_{nn}^r)^2 + 2 \sum_{i < j} (h_{ij}^r)^2] \\
 &\quad - 2 \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} h_{ii}^r h_{jj}^r - n(n-1) \frac{c+3}{4} - [3(n-1) \cos^2 \theta - 2n + 2] \frac{c-1}{4} \\
 &= 2\tau + \frac{1}{2} \sum_{r=n+1}^{2m+1} [(h_{11}^r + \dots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \dots - h_{nn}^r)^2] + 2 \sum_{r=n+1}^{2m+1} \sum_{i < j} (h_{ij}^r)^2 \\
 &\quad - 2 \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} h_{ii}^r h_{jj}^r - n(n-1) \frac{c+3}{4} - [3(n-1) \cos^2 \theta - 2n + 2] \frac{c-1}{4}.
 \end{aligned}$$

From the equation of Gauss, we find

$$K_{ij} = \sum_{r=n+1}^{2m+1} [h_{11}^r h_{22}^r - (h_{12}^r)^2] + 3 \cos^2 \theta \cdot \frac{c-1}{4} + \frac{c+3}{4}$$

and consequently

$$\begin{aligned}
 (2.4) \quad \sum_{2 \leq i < j \leq n} K_{ij} &= \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2} \frac{c+3}{4} + \\
 &\quad + [3(n-1) \cos^2 \theta - 3 \cos^2 \theta - 2n + 4] \frac{c-1}{8}.
 \end{aligned}$$

Substituting (2.4) in (2.3), one gets

$$\frac{1}{2} n^2 \|H\|^2 \geq 2 \operatorname{Ric}(X) - 2(n-1) \frac{c+3}{4} - (3 \cos^2 \theta - 2) \frac{c-1}{4},$$

which is equivalent to (2.1).

(ii) Assume $H(p) = 0$. Equality holds in (2.1) if and only if

$$(2.5) \quad \begin{cases} h_{12}^r = \dots = h_{1n}^r = 0, \\ h_{11}^r = h_{22}^r + \dots + h_{nn}^r, r \in \{n+1, \dots, 2m\}. \end{cases}$$

Then $h_{1j}^r = 0$, for every $j \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m\}$, that is $X \in \mathcal{N}_p$.

(iii) The equality case of (2.1) holds for all unit tangent vectors orthogonal to ξ at p if and only if

$$(2.6) \quad \begin{cases} h_{ij}^r = 0, i \neq j, r \in \{n+1, \dots, 2m\}, \\ h_{11}^r + \dots + h_{nn}^r - 2h_{ii}^r = 0, \quad i \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m\}. \end{cases}$$

In this case, since ξ is tangent to M , it follows that a totally umbilical point is totally geodesic. □

Theorem 2.2. *Let M be an $(n = 2d_1 + 2d_2 + 1)$ -dimensional bi-slant submanifold satisfying $g(X, \phi Y) = 0$, for any $X \in \mathcal{D}_1$ and any $X \in \mathcal{D}_2$, tangent to ξ in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$. Then:*

- (i) For each unit vector $X \in T_p M$ orthogonal to ξ and if
 - a) X is tangent to \mathcal{D}_1 we have

$$(2.7) \quad \operatorname{Ric}(X) \leq \frac{1}{4} \{ (n-1)(c+3) + \frac{1}{2} (3 \cos^2 \theta_1 - 2)(c-1) + n^2 \|H\|^2 \}$$

and if

b) X is tangent to \mathcal{D}_2 we have

$$(2.7') \quad \text{Ric}(X) \leq \frac{1}{4}\{(n-1)(c+3) + \frac{1}{2}(3 \cos^2 \theta_2 - 2)(c-1) + n^2 \|H\|^2\}.$$

(ii) If $H(p) = 0$, then a unit tangent vector $X \in T_p M$ orthogonal to ξ satisfies the equality case of (2.7) and (2.7') if and only if $X \in \mathcal{N}_p$.

(iii) The equality case of (2.7) and (2.7') holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.

Proof. Let $X \in T_p M$ be a unit tangent vector X at p , orthogonal to ξ . We choose an orthonormal basis $e_1, \dots, e_n = \xi, e_{n+1}, \dots, e_{2m+1}$ such that e_1, \dots, e_n are tangent to M at p , with $e_1 = X$.

Then, from the equation of Gauss, we have

$$(2.8) \quad n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1) \frac{c+3}{4} - [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n+2] \frac{c-1}{4}.$$

From (2.8), we get

$$(2.9) \quad n^2 \|H\|^2 = 2\tau + \sum_{r=n+1}^{2m+1} [(h_{11}^r)^2 + (h_{22}^r + \dots + h_{nn}^r)^2 + 2 \sum_{i<j} (h_{ij}^r)^2] - 2 \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} h_{ii}^r h_{jj}^r - n(n-1) \frac{c+3}{4} - [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n+2] \frac{c-1}{4} =$$

$$= 2\tau + \frac{1}{2} \sum_{r=n+1}^{2m+1} [(h_{11}^r + \dots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \dots - h_{nn}^r)^2] + 2 \sum_{r=n+1}^{2m+1} \sum_{i<j} (h_{ij}^r)^2 - 2 \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} h_{ii}^r h_{jj}^r - n(n-1) \frac{c+3}{4} - [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 2n+2] \frac{c-1}{4}.$$

From the equation of Gauss, we find:

a) if X is tangent to \mathcal{D}_1

$$K_{ij} = \sum_{r=n+1}^{2m+1} [h_{11}^r h_{22}^r - (h_{12}^r)^2] + 3 \cos^2 \theta_1 \cdot \frac{c-1}{4} + \frac{c+3}{4}$$

and consequently

$$(2.10) \quad \sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2} \frac{c+3}{4} + [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 3 \cos^2 \theta_1 - 2n+4] \frac{c-1}{8}.$$

Substituting (2.10) in (2.9), one gets

$$\frac{1}{2} n^2 \|H\|^2 \geq 2 \text{Ric}(X) - 2(n-1) \frac{c+3}{4} - (3 \cos^2 \theta_1 - 2) \frac{c-1}{4},$$

which is equivalent to (2.7).

b) Similar if X is tangent to \mathcal{D}_2 , we have

$$K_{ij} = \sum_{r=n+1}^{2m+1} [h_{11}^r h_{22}^r - (h_{12}^r)^2] + 3 \cos^2 \theta_2 \cdot \frac{c-1}{4} + \frac{c+3}{4}$$

and consequently

$$(2.11) \quad \sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2} \frac{c+3}{4} + [6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - 3 \cos^2 \theta_2 - 2n + 4] \frac{c-1}{8}.$$

Substituting (2.11) in (2.9), one gets

$$\frac{1}{2} n^2 \|H\|^2 \geq 2 \operatorname{Ric}(X) - 2(n-1) \frac{c+3}{4} - (3 \cos^2 \theta_2 - 2) \frac{c-1}{4},$$

which is equivalent to (2.7').

(ii) Assume $H(p) = 0$. Equality holds in (2.7) and (2.7') if and only if

$$(2.12) \quad \begin{cases} h_{12}^r = \dots = h_{1n}^r = 0, \\ h_{11}^r = h_{22}^r + \dots + h_{nn}^r, r \in \{n+1, \dots, 2m\}. \end{cases}$$

Then $h_{1j}^r = 0$, for every $j \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m\}$, that is $X \in \mathcal{N}_p$.

(iii) The equality case of (2.7) and (2.7') holds for all unit tangent vectors orthogonal to ξ at p if and only if

$$(2.13) \quad \begin{cases} h_{ij}^r = 0, i \neq j, r \in \{n+1, \dots, 2m\}, \\ h_{11}^r + \dots + h_{nn}^r - 2h_{ii}^r = 0, \quad i \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m\}. \end{cases}$$

In this case, since ξ is tangent to M , it follows that a totally umbilical point is totally geodesic. □

Corollary 2.3. *Let M be an $(n = 2d_1 + 2d_2 + 1)$ -dimensional semi-slant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$. Then:*

(i) *For each unit vector $X \in T_p M$ orthogonal to ξ and if a X is tangent to \mathcal{D}_1 we have*

$$(2.14) \quad \operatorname{Ric}(X) \leq \frac{1}{4} \{(n-1)(c+3) - (c-1) + n^2 \|H\|^2\}$$

and if

b X is tangent to \mathcal{D}_2 we have

$$(2.14') \quad \operatorname{Ric}(X) \leq \frac{1}{4} \{(n-1)(c+3) + \frac{1}{2}(3 \cos^2 \theta - 2)(c-1) + n^2 \|H\|^2\}.$$

(ii) *If $H(p) = 0$, then a unit tangent vector $X \in T_p M$ orthogonal to ξ satisfies the equality case of (2.14) and (2.14') if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of (2.14) and (2.14') holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Corollary 2.4. *Let M be an $(n = 2k + 1)$ -dimensional invariant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$. Then:*

(i) *For each unit vector $X \in T_p M$ orthogonal to ξ , we have*

$$(2.15) \quad \operatorname{Ric}(X) \leq \frac{1}{4} \{(n-1)(c+3) + \frac{1}{2}(c-1)\}.$$

(ii) *A unit tangent vector $X \in T_p M$ orthogonal to ξ satisfies the equality case of (2.15) if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of (2.15) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Corollary 2.5. *Let M be an $(n = 2k + 1)$ -dimensional anti-invariant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$. Then:*

(i) *For each unit vector $X \in T_p M$ orthogonal to ξ , we have*

$$(2.16) \quad \text{Ric}(X) \leq \frac{1}{4}\{(n-1)(c+3) - (c-1) + n^2\|H\|^2\}.$$

(ii) *If $H(p) = 0$, then a unit tangent vector $X \in T_p M$ orthogonal to ξ satisfies the equality case of (2.16) if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of (2.16) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

3. k -RICCI CURVATURE

In this section, we prove a relationship between the k -Ricci curvature and the squared mean curvature for slant, bi-slant and semi-slant submanifolds in a Sasakian space form.

We state an inequality between the scalar curvature and the squared mean curvature for submanifolds tangent to ξ .

Theorem 3.1. *Let M be an $(n = 2k + 1)$ -dimensional θ -slant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ . Then we have*

$$(3.1) \quad \|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{[3(n-1)\cos^2\theta - 2n+2](c-1)}{4n(n-1)}.$$

Proof. We choose an orthonormal basis $\{e_1, \dots, e_n = \xi, e_{n+1}, \dots, e_{2m+1}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and e_1, \dots, e_n diagonalize the shape operator A_{n+1} . Then the shape operators take the forms

$$(3.2) \quad A_{n+1} = \begin{pmatrix} a_1 & 0 & \dots & \dots & 0 \\ 0 & a_2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & a_n \end{pmatrix}$$

$$(3.3) \quad A_r = (h_{ij}^r), i, j = 1, \dots, n, r = n+2, \dots, 2m+1, \text{trace } A_r = \sum_{i=1}^n h_{ii}^r = 0.$$

From (2.2), we get

$$(3.4) \quad n^2\|H\|^2 = 2\tau + \sum_{i=1}^n a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^n (h_{ij}^r)^2 - n(n-1)\frac{c+3}{4} - [3(n-1)\cos^2\theta - 2n+2]\frac{c-1}{4}.$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j,$$

we obtain

$$n^2\|H\|^2 = \left(\sum_{i=1}^n a_i\right)^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^n a_i^2,$$

which implies

$$\sum_{i=1}^n a_i^2 \geq n\|H\|^2.$$

Since we have that

$$(3.5) \quad n^2 \|H\|^2 \geq 2\tau + n \|H\|^2 - n(n-1) \frac{c+3}{4} - [3(n-1) \cos^2 \theta - 2n + 2] \frac{c-1}{4},$$

which is equivalent to (3.1).

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_p M$. Denote by $L_{i_1 \dots i_k}$ the k -plane section spanned by e_{i_1}, \dots, e_{i_k} . It follows from (1.7) and (1.8) that

$$(3.6) \quad \tau(L_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} \text{Ric}_{L_{i_1 \dots i_k}}(e_i),$$

$$(3.7) \quad \tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(L_{i_1 \dots i_k}).$$

Combining (1.9), (3.6) and (3.7), we find

$$(3.8) \quad \tau(p) \geq \frac{n(n-1)}{2} \Theta_k(p).$$

□

From (3.6), (3.7) and (3.1), we get the following.

Theorem 3.2. *Let M be an $(n = 2d_1 + 2d_2 + 1)$ -dimensional bi-slant submanifold satisfying $g(X, \phi Y) = 0$, for any $X \in \mathcal{D}_1$ and any $X \in \mathcal{D}_2$, in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ . Then we have*

$$(3.9) \quad \|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{[3(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - n + 1](c-1)}{2n(n-1)}.$$

Proof. The proof is similar with their corresponding statements of Theorem 3.1. □

Theorem 3.3. *Let M be an $(n = 2d_1 + 2d_2 + 1)$ -dimensional semi-slant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ . Then we have*

$$(3.10) \quad \|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{[3(d_1 + d_2 \cos^2 \theta) - n + 1](c-1)}{2n(n-1)}.$$

Proof. The proof is similar with their corresponding statements of Theorem 3.1. □

Theorem 3.4. *Let M be an $(n = 2k + 1)$ -dimensional θ -slant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ . Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$(3.11) \quad \|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(n-1) \cos^2 \theta - 2n + 2](c-1)}{4n(n-1)}.$$

Theorem 3.5. *Let M be an $(n = 2d_1 + 2d_2 + 1)$ -dimensional bi-slant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ . Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$(3.12) \quad \|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) - n + 1](c-1)}{2n(n-1)}.$$

Theorem 3.6. *Let M be an $(n = 2d_1 + 2d_2 + 1)$ -dimensional semi-slant submanifold in a $(2m + 1)$ -dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ . Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$(3.13) \quad \|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(d_1 + d_2 \cos^2 \theta) - n + 1](c-1)}{2n(n-1)}.$$

Corollary 3.7. *Let M be an n -dimensional invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$(3.14) \quad \Theta_k(p) \leq \frac{c+3}{4} + \frac{c-1}{4n}.$$

Corollary 3.8. *Let M be an n -dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$(3.15) \quad \|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} + \frac{c-1}{2n}.$$

Corollary 3.9. *Let M be an n -dimensional contact CR submanifold ($\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$) of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$(3.16) \quad \|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{(3d_1 - n + 1)(c-1)}{2n(n-1)}.$$

where $2d_1 = \dim \mathcal{D}_1$.

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