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TRANSLATION INVARIANT OPERATORS ON HARDY SPACES OVER VILENKIN GROUPS

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Dedicated to Professor William R. Wade on the occasion of his 60th birthday

Abstract. We show that a number of well known multiplier theorems for Hardy spaces over Vilenkin groups follow immediately from a general condition on the kernel of the multiplier operator. In the compact case, this result shows that the multiplier theorems of Kitada [6], Tateoka [13], Daly-Phillips [2], and Simon [11] are best viewed as providing conditions on the partial sums of the Fourier-Vilenkin series of the kernel rather than explicit conditions on the Fourier-Vilenkin coefficients themselves. The theorem is used to prove an extension of the Marcinkiewicz multiplier theorem for Hardy spaces.

1. Introduction

In this paper the setting will be a locally compact Vilenkin group G of bounded order. Thus G contains a decreasing sequence of compact open subgroups $(G_n)_{n=-\infty}^{\infty}$ such that

i) $\bigcup_{-\infty}^{\infty} G_n = G$ and $\bigcap_{-\infty}^{\infty} G_n = \{0\}$,
ii) $\sup_n\{\text{order}(G_n/G_{n+1})\} < \infty$.

In the case that G is compact, we use the convention that $G_n = G$ if $n \leq 0$. The additive group of a local field is Vilenkin group, as is its ring of integers. In particular, the *p-adic* numbers are a Vilenkin group. In the case that $p = 2$, the ring of integers is also called the dyadic group and the characters the Walsh functions.

Let Γ denote the dual group of G and $\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}$. The Haar measures μ on G and λ on Γ are chosen so that $\mu(G_0) = \lambda(\Gamma_0) = 1$ and consequently, $\mu(G_n) = (\lambda(\Gamma_n))^{-1} := (M_n)^{-1}$ for each $n \in Z$. There is a norm on G defined by $|x| = (M_n)^{-1}$ if $x \in G_n \backslash G_{n+1}$. The Fourier transform and inverse Fourier transform respectively are denoted by $\wedge$ and $\vee$, and satisfy

$$(\xi_{G_n})^{\wedge} = (\lambda(\Gamma_n))^{-1} \xi_{\Gamma_n}$$

where ξ_A denotes the characteristic function of a set A. Consequently,

$$(\xi_{\Gamma_n})^{\vee} = (\lambda(G_n))^{-1} \xi_{G_n}.$$

We define distributions according to the theory developed by Taibleson [12] for local fields. Let $S(G)$ be defined as the collection of functions that have compact support and that are constants on the cosets of a G_n $(n \in \mathbb{Z})$. A sequence (ψ_k) in $S(G)$ is said to converge to $\psi \in S(G)$ if there are $n, m \in \mathbb{Z}$ such that every ψ_k is

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constant on the cosets of G_m, $\operatorname{supp}\psi_k \subset G_n$ $(k \in \mathbb{N})$, and (ψ_k) converges uniformly to ψ. Continuous linear functionals on $S(G)$ are called distributions. The set of distributions will be denoted by $S'(G)$.

The (atomic) Hardy spaces on G are given as follows. A function $a\colon G \to C$ is a p-atom, $0 < p \leq 1$, if

i) $\operatorname{supp} a \subset I_n := x + G_n$ for some $x \in G$, and $n \in \mathbb{N}$,

ii) $\|a\|_\infty \leq (\mu(I_n))^{-1/p}$,

iii) $\int_G a(x)dx = 0$.

A distribution $f \in S'(G)$ belongs to $H^p(G)$ if f is given by $f = \sum_{i=1}^\infty \lambda_i a_i$, where each a_i is a p-atom, $\sum_{i=1}^\infty |\lambda_i|^p < \infty$, and convergence is in $S'(G)$. We set

$$\|f\|_{H^p} = \inf \left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{1/p}$$

with the infimum taken over all such atomic decompositions of f. A function $\varphi \in L^\infty(\Gamma)$ is a (Fourier) multiplier on $H^p(G)$ if there exists a constant $C > 0$ so that for all $f \in H^p(G) \cap L^2(G)$,

$$\left\|(\varphi f^\wedge)^\vee\right\|_{H^p} \leq C\,\|f\|_{H^p}.$$

A multiplier operator T_φ is defined for a function φ on Γ by

$$(T_\varphi f)^\wedge = \varphi \cdot f^\wedge.$$

The operator T_φ is a convolution operator determined by the distribution Φ which has kernel $\Bbbk$ defined by

$$\Bbbk^\wedge = \varphi.$$

The blocks $\Delta_n \Bbbk$ of the kernel $\Bbbk$ are defined by $\Delta_n \Bbbk = (\Bbbk^\wedge \xi_{\Gamma_{n+1}\backslash\Gamma_n})^\vee$ $(n \in \mathbb{Z})$. For a multiplier φ, the blocks are $\Delta_n\varphi = \varphi\,\xi_{\Gamma_{n+1}\backslash\Gamma_n}$.

2. Results and proofs

A number of authors have proved multiplier theorems for $H^p(G)$. Among them are Daly, Fridli, Kitada, Onneweer, Phillips, Quek, Simon, and Tateoka. The results of Kitada [6], Onneweer-Quek [8], and Tateoka [13] often were phrased in terms of blocks of the kernel belonging to certain Herz spaces along with growth bounds. These were called multiplier theorems; even though, the theorems are most naturally phrased in terms of the corresponding kernel.

First we formulate Theorem 1 which is a general result for a convolution operator with kernel $\Bbbk$ to be a bounded operator on $H^p(G)$. Then we formulate Theorem 2. From this theorem we will show that all of the previous multiplier results follow in a straight forward manner. Finally, we will use it to prove an $H^p(G)$ version of the classical Marcinkiewicz multiplier theorem.

Theorem 1. *Let $\Bbbk$ be locally integrable on $G\backslash\{0\}$ and $0 < p \leq 1$. If either*

i) $\sup_N \int_{(G_N)^c} |G_N|^{-1} \left(\int_{G_N} |\Bbbk(x-y)|\,dy\right)^p dx < \infty$

or

ii) $\sup_N \int_{(G_N)^c} |G_N|^{-1} \left(\int_{G_N} |\Bbbk(x-y) - \Bbbk(x)|\,dy\right)^p dx < \infty,$

then $T_\Bbbk$ is bounded on $H^p(G)$.

Theorem 1 in the case of $p = 1$ has appeared many places in the literature. For example, Inglis [4] proves a version for totally disconnected groups and a version for local fields appears in the paper of Phillips and Taibleson [9]. In both examples, they were concerned with boundedness questions of operators on L^r, $1 < r < \infty$, and weak(L^1) results. As the atomic theory of Hardy spaces was developed, these results were extended to H^1. See [5] for an example.

If the kernel $\mathbb{k}$ is decomposed into blocks then one can get the following sufficient conditions that turned to be useful in applications.

Theorem 2. *Let $\mathbb{k}$ be locally integrable on $G\backslash\{0\}$ and $0 < p \leq 1$. If either*

i) $\sup_N \int_{(G_N)^c} |G_N|^{-1} \left(\int_{G_N} |\sum_{j=N+1}^{\infty} \Delta_j\mathbb{k}(x-y)|\, dy\right)^p dx < \infty$

or

ii) $\sup_N \int_{(G_N)^c} |G_N|^{-1} \left(\int_{G_N} |\sum_{j=N+1}^{\infty} (\Delta_j\mathbb{k}(x-y) - \Delta_j\mathbb{k}(x))|\, dy\right)^p dx < \infty,$

then $T_{\mathbb{k}}$ is bounded on $H^p(G)$.

Condition ii) of Theorem 1 and Theorem 2 is useful in analyzing the boundedness properties of singular integral type operators. For example, in the case of **q**-series or **q**-adic fields K_q, Calderon-Zygmund singular integral operators have been studied extensively. See Phillips-Taibleson [9] for the $L^p(K_{\mathbf{q}})$, $1 < p < \infty$, case and Daly-Phillips [3] for the $H^p(K_q), 0 < p \leq 1$, case. These operators have homogeneity in the kernels $\mathbb{k}$: $\mathbb{k}(\mathbf{q}^j x) = q^{-j}\mathbb{k}(x)$. Thus the kernel can be written as $\mathbb{k} = \omega \bullet |\cdot|^{-1}$ with $\omega(\mathbf{q}^j x) = \omega(x)$ for $x \neq 0$. The kernel $\mathbb{k}$ is said to be homogeneous of degree -1. If the kernel satisfies

$$\int_{|y|\leq 1} \int_{|x|>1} |\mathbb{k}(x-y) - \mathbb{k}(x)|\, dxdy < \infty$$

then $T_{\mathbb{k}}$ is bounded on $L^p(K_{\mathbf{q}})$ for $1 < p < \infty$ and $H^1(K_{\mathbf{q}})$ (see [3]). Using the homogeneity of the kernel, this condition is easily seen to be equivalent to our condition ii) of Theorem 1 for $p = 1$. Also, if one chooses to decompose the kernel into blocks in a manner inconsistent with the subgroup decompositions of Γ, then one would begin the proof of boundedness using Theorem 1 directly and not use Theorem 2. For example, Wo-Sang Young does so in [15] where she proves a Marcinkiewicz multiplier theorem using dyadic blocks for an arbitrary compact Vilenkin group.

We proceed with listing conditions that are sufficient for the multiplier operator be bounded on $H^p(G)$, and that have been used by several authors. They all can be considered as consequences of Theorem 1.

Corollary 3. If $\mathbb{k}$ is locally integrable on $G\backslash\{0\}$ and $0 < p \leq 1$, and

$$\sup_N \sum_{j=N+1}^{\infty} \int_{(G_N)^c} |G_N|^{-1} \Big(\int_{G_N} |\Delta_j\mathbb{k}(x-y)|\, dy\Big)^p dx < \infty,$$

then $T_{\mathbb{k}}$ is bounded on $H^p(G)$.

We note that this condition was used by Simon [11] in the special case when G is a compact bounded multiplicative Vilenkin group. He sated the result in terms of $(\Delta_j\varphi)^\vee$ rather than $\Delta_j\mathbb{k}$.

In the following corollary we assume that $p = 1$. It was first formalized and used by Kitada [5] and Tateoka [13].

Corollary 4. Let $\mathbb{k}$ be locally integrable on $G\backslash\{0\}$ and $0 < p \leq 1$. If

$$\sup_N \int_{(G_N)^c} \sum_{j=-\infty}^{\infty} |(\Delta_j\mathbb{k})(x)|\, dx < \infty,$$

then $T_{\mathbb{k}}$ is bounded on $H^1G)$.

Daly and Phillips [2] observed that the condition in Corollary 4 can be relaxed. Namely, they proved that it is enough to start the summation from $N+1$ instead of $-\infty$.

Corollary 5. Let $\Bbbk$ be locally integrable on $G\backslash\{0\}$ and $0<p\le 1$. If

$$\sup_N \int_{(G_N)^c} \sum_{j=N+1}^{\infty} |(\Delta_j \Bbbk)(x)|\, dx < \infty,$$

then $T_\Bbbk$ is bounded on $H^1(G)$.

The condition in the following corollary is due to Kitada [5] and Tateoka [13]. We note that it was used for example by Daly and Fridli in [1] for Walsh multipliers.

Corollary 6. Let $\Bbbk$ be locally integrable on $G\backslash\{0\}$ and $0<p\le 1$. If

$$\sum_{N=-\infty}^{j} |G_N|^{1-p}\Big(\int_{G_N\backslash G_{N+1}} |\Delta_j \Bbbk(y)|\, dy\Big)^p \le C|G_j|^{1-p},$$

then $T_\Bbbk$ is bounded on $H^p(G)$.

We will first provide the proofs of the corollaries, assuming Theorem 2, and then provide the proof of Theorem 1 and Theorem 2. For Corollary 3, we use i) from Theorem 2 and the fact $p\le 1$:

$$\begin{aligned}\int_{(G_N)^c} &|G_N|^{-1}\Big(\int_{G_N} |\sum_{j=N+1}^{\infty} \Delta_j \Bbbk(x-y)|\, dy\Big)^p dx\\ &\le \int_{(G_N)^c} |G_N|^{-1}\Big(\int_{G_N} \sum_{j=N+1}^{\infty} |\Delta_j \Bbbk(x-y)|\, dy\Big)^p dx\\ &\le \sum_{j=N+1}^{\infty} \int_{(G_N)^c} |G_N|^{-1}\Big(\int_{G_N} |\Delta_j \Bbbk(x-y)|\, dy\Big)^p dx.\end{aligned}$$

Taking the supremum over N, we obtain Corollary 3.

To prove Corollary 5 with the condition of Daly and Phillips [2] for $H^1(G)$, we proceed from Corollary 3 with $p=1$:

$$\sum_{j=N+1}^{\infty} \int_{(G_N)^c} |G_N|^{-1} \int_{G_N} |\Delta_j \Bbbk(x-y)|\, dy\, dx\,.$$

As $x\in (G_N)^c$, $y\in G_N$ we have that the value inner integral does not actually depend on y. Indeed, $\int_{G_N} |\Delta_j \Bbbk(x-y)|\, dy\, dx = \int_{x+G_N} |\Delta_j \Bbbk(t)|\, dt$. The function $|G_N|^{-1}\int_{x+G_N} |\Delta_j \Bbbk(t)|\, dt$ is nothing but the integral average function of $|\Delta_j \Bbbk|$ over the cosets of G_n. Consequently it is constant on these cosets and its integral over $(G_N)^c$ is equal to the integral of the function, i.e.

$$\sum_{j=N+1}^{\infty} \int_{(G_N)^c} |G_N|^{-1} \int_{G_N} |\Delta_j \Bbbk(x-y)|\, dy\, dx \,=\, \sum_{j=N+1}^{\infty} \int_{(G_N)^c} |\Delta_j \Bbbk(t)|\, dt.$$

Thus the condition in Corollary 3 and the Daly-Phillips conditions coincide when $p=1$. Allowing the above sum to run from $-\infty$ to ∞, one obtains the Kitada-Tateoka ([6], [13]) condition, i.e. Corollary 4 for $H^1(G)$.

Applying the same argument to condition from Corollary 3 for $0<p<1$, and a Hölder inequality with exponent $1/p$ we obtain the following condition.

Corollary 7. Let $\Bbbk$ be locally integrable on $G\backslash\{0\}$ and $0 < p \leq 1$. Then

$$\sup_N \sum_{j=N+1}^{\infty} |G_N|^{p-1}\Big(\int_{(G_N)^c} |(\Delta_j \Bbbk(t)|\, dt\Big)^p < \infty$$

implies that the operator $T_\Bbbk$ is bounded on $H^p(G)$.

The proof of Corollary 6 for $H^p(G)$ is more involved than the previous. Beginning again with $i)$ from Theorem 2:

$$\begin{aligned} U_N :=& \int_{(G_N)^c} |G_N|^{-1}\Big(\int_{G_N} |\sum_{j=N+1}^{\infty} \Delta_j \Bbbk(x-y)|\, dy\Big)^p dx \\ =& \sum_{n=-\infty}^{N-1} \int_{G_n\backslash G_{n+1}} |G_N|^{-1}\Big(\int_{G_N} |\sum_{j=N+1}^{\infty} \Delta_j \Bbbk(x-y)|\, dy\Big)^p dx \\ \leq& |G_N|^{-1} \sum_{n=-\infty}^{N-1} \int_{G_n\backslash G_{n+1}} \Big(\int_{G_N} \sum_{j=N+1}^{\infty} |\Delta_j \Bbbk(x-y)|\, dy\Big)^p dx. \end{aligned}$$

Using the Hölder inequality on the outer integral with $r = 1/p$ and $r' = 1/(1-p)$, we continue with

$$\begin{aligned} U_N \leq& |G_N|^{-1} \sum_{n=-\infty}^{N-1} \Big(\int_{G_n\backslash G_{n+1}} \int_{G_N} \sum_{j=N+1}^{\infty} |\Delta_j \Bbbk(x-y)|\, dy\, dx\Big)^p \\ & \qquad \times \Big(\int_G \xi_{G_n\backslash G_{n+1}}(y) dy\Big)^{1-p} \\ \leq& |G_N|^{-1} \sum_{n=-\infty}^{N-1} |G_n|^{1-p}\Big(\int_{G_n\backslash G_{n+1}} \int_{G_N} \sum_{j=N+1}^{\infty} |\Delta_j \Bbbk(x-y)|\, dy\, dx\Big)^p. \end{aligned}$$

Making use of the fact that $x - y \in G_n\backslash G_{n+1}$ when $N > n$, $y \in G_N$, and $x \in G_n$, we have $\int_{G_N} |\Delta_j \Bbbk(x-y)|\, dy = \int_{x+G_N} |\Delta_j \Bbbk(t)|\, dt$. Therefore the inequality becomes

$$\begin{aligned} U_N \leq& |G_N|^{p-1} \sum_{n=-\infty}^{N-1} |G_n|^{1-p} \Big(\sum_{j=N+1}^{\infty} \int_{G_n\backslash G_{n+1}} |\Delta_j \Bbbk(x)|\, dx\Big)^p \\ \leq& |G_N|^{p-1} \sum_{j=N+1}^{\infty} \sum_{n=-\infty}^{N-1} |G_n|^{1-p}\Big(\int_{G_n\backslash G_{n+1}} |\Delta_j \Bbbk(x)|\, dx\Big)^p. \end{aligned}$$

Since $j \geq N+1$ we have that the inner sum can be estimated above by the left side of condition from Corollary 6. It is bounded by $C|G_j|^{1-p}$. Thus

$$U_N \leq C|G_N|^{p-1} \sum_{j=N+1}^{\infty} |G_j|^{1-p} \leq C|G_N|^{p-1}|G_N|^{1-p} = C.$$

We now proceed with the proof of Theorem 1 and Theorem 2.

Proofs of Theorems 1 and 2. We note that it is sufficient to show $T_\Bbbk(a) \in L^p(G)$. Without the loss of generality we may suppose that $\operatorname{supp} a \subset G_N$, $\|a||_{L^\infty(G)} \leq |G_N|^{-1/p}$, and $\int_{G_N} a = 0$. Set

$$(1) \qquad \|T_\Bbbk(a)\|^p_{L^p(G)} = \int_{G_N} |T_\Bbbk(a)(x)|^p dx + \int_{(G_N)^c} |T_\Bbbk(a)(x)|^p dx = T_1 + T_2.$$

For T_1 we use the usual L^2 argument that exploits the facts that $T_{\Bbbk}$ is bounded on L^2 and $a \in L^2$:

$$\begin{aligned}
T_1 &= \int_G |T_{\Bbbk}(a)(x)|^p \, \xi_{G_N}(x)dx \\
&\le \Big(\int_G |T_{\Bbbk}(a)(x)|^2 dx \Big)^{\frac{p}{2}} \Big(\int_G \xi_{G_N}(x)dx \Big)^{1-\frac{p}{2}} \\
&\le C\|a\|_2^p |G_N|^{1-\frac{p}{2}} \\
&\le C|G_N|^{(\frac{1}{2}-\frac{1}{p})p}|G_N|^{1-\frac{p}{2}} \\
&= C.
\end{aligned}$$

For T_2 we will use the boundedness and cancellation properties of the atom a. One direction is

$$T_2 = \int_{(G_N)^c} \Big| \int_{G_N} \Bbbk(x-y)a(y)dy \Big|^p dx \le \int_{(G_N)^c} |G_N|^{-1} \Big(\int_{G_N} |\Bbbk(x-y)| \, dy \Big)^p dx$$

and the other is

$$\begin{aligned}
T_2 &= \int_{(G_N)^c} \Big| \int_{G_N} \Bbbk(x-y)a(y)\, dy \Big|^p dx \\
&= \int_{(G_N)^c} \Big| \int_{G_N} (\Bbbk(x-y) - \Bbbk(x))a(y)\, dy \Big|^p dx \\
&\le \int_{(G_N)^c} |G_N|^{-1} \Big(\int_{G_N} \Big| (\Bbbk(x-y) - \Bbbk(x)) \Big| \, dy \Big)^p dx.
\end{aligned}$$

This proves Theorem 1.

Let us take (1) again. To prove Theorem 2 we decompose the kernel $\Bbbk$ in terms of the blocks of its Fourier-Vilenkin transform $\Bbbk = \sum_{j=-\infty}^{\infty} \Delta_j \Bbbk$. Using this decomposition, T_2 becomes in the first case

$$\begin{aligned}
\int_{(G_N)^c} \Big| \int_{G_N} \Bbbk(x-y)a(y) \Big|^p dx \le \int_{(G_N)^c} \Big(\Big| \int_{G_N} \sum_{j=-\infty}^{N-1} \Delta_j \Bbbk(x-y)a(y)\, dy \Big| dx \\
+ \Big| \int_{G_N} \sum_{j=N}^{\infty} \Delta_j \Bbbk(x-y)a(y)\, dy \Big| \Big)^p dx \,.
\end{aligned}$$

Since $\Delta_j \Bbbk(x-y) = \Delta_j \Bbbk(x)$ as $j < N$ and $y \in G_N$, and using the fact $\int_{G_N} a = 0$, we have that the first integrand is identically zero. Combining this with our estimates for T_1

$$\|T_{\Bbbk}(a)\|_{L^p(G)}^p \le C + \int_{(G_N)^c} \Big(\Big| \int_{G_N} \sum_{j=N}^{\infty} \Delta_j \Bbbk(x-y)a(y)\, dy \Big| \Big)^p dx = C + U_1.$$

Using again the fact $\int_{G_N} a = 0$, U_1 can be rewritten as

$$U_2 = \int_{(G_N)^c} \Big(\Big| \int_{G_N} \sum_{j=N}^{\infty} (\Delta_j \Bbbk(x-y) - \Delta_j \Bbbk(x))a(y)\, dy \Big| \Big)^p dx.$$

The final estimates for both U_1 and U_2 follow from $\|a\|_{L^\infty(G)} \le |G_N|^{-1/p}$. Indeed, for U_1 we have

$$\begin{aligned}
U_1 &\le \int_{(G_N)^c} \Big(\int_{G_N} \Big| \sum_{j=N}^{\infty} \Delta_j \Bbbk(x-y) \Big| |a(y)|\, dy \Big)^p dx \\
&\le \int_{(G_N)^c} |G_N|^{-1} \Big(\int_{G_N} \Big| \sum_{j=N}^{\infty} \Delta_j \Bbbk(x-y) \Big| \, dy \Big)^p dx.
\end{aligned}$$

This is the required estimate for (i) of Theorem 2. As stated above, the estimate for (ii) of Theorem 2 is is obtained in an identical manner from U_2. $\square$

We will use Theorem 2 in the form of Corollary 5 (Kitada, Tateoka) to prove a version of the Marcinkiewicz multiplier theorem for $H^p(G)$. This will be for the compact multiplicative G. Then the dual group $\Gamma = \{\chi_n\}$ can be enumerated in the way that corresponds to the Paley enumeration in the Walsh case. The Dirichlet kernels are defined as $D_n = \sum_{k=0}^{n-1} \chi_k \quad (n \in \mathbb{N})$. For details we refer the reader to [10].

First we will need a lemma that is a type of Sidon inequality. The authors [1] earlier proved a version for the dyadic group and Walsh series.

Lemma 8. *Let G be compact multiplicative Vilenkin group. If $n, N \in \mathbb{N}$ and $1 < q \le 2$ then for any numbers c_k $(1 \le k \le |\Gamma_n|)$, we have*

$$\int_{(G_N)^c} \Big| \sum_{k=1}^{|\Gamma_n|} c_k D_k(x) \Big| \, dx \le C |G_N|^{\frac{1}{q}-1} \Big(\sum_{k=1}^{|\Gamma_n|} |c_k|^q \Big)^{1/q}.$$

Proof. The generalized Rademacher functions, see e.g. [10] for the definition, will be denoted by r_j $(j \in \mathbb{N})$. By means of the Rademacher function the Dirichlet kernels can be decomposed as $D_k = \chi_k \sum_{j=0}^{\infty} \sum_{\ell=m_j-k_j}^{m_j-1} r_j^\ell D_{|\Gamma_j|}$ ([10]). We note that $D_{|\Gamma_n|} = |G_N|^{-1} \xi_{G_N}$ ([10]).

Without loss of generality, we may assume $n > N$. Then

$$\begin{aligned} \int_{(G_N)^c} \Big| \sum_{k=1}^{|\Gamma_n|} c_k D_k(x) \Big| \, dx &= \int_{(G_N)^c} \Big| \sum_{k=1}^{|\Gamma_n|} c_k \chi_k(x) \sum_{j=0}^{\infty} \sum_{\ell=m_j-k_j}^{m_j-1} r_j^\ell D_{|\Gamma j|}(x) \Big| \, dx \\ &= \int_{(G_N)^c} \Big| \sum_{k=1}^{|\Gamma_n|} c_k \chi_k(x) \sum_{j=0}^{N-1} \sum_{\ell=m_j-k_j}^{m_j-1} r_j^\ell D_{|\Gamma j|}(x) \Big| \, dx \\ &\le \sum_{j=0}^{N-1} |G_j|^{-1} \int_{(G_N)^c} \xi_{G_j}(x) \Big| \sum_{\ell=m_j-k_j}^{m_j-1} r_j^\ell \sum_{k=1}^{|\Gamma_n|} c_k \chi_k(x) \Big| \, dx. \end{aligned}$$

Set

$$k_{j,\ell} = \begin{cases} 1 & \text{if,} \quad m_j - k_j \le \ell \le m_j - 1 \\ 0 & \text{if,} \quad 0 \le \ell < m_j - k_j \end{cases} \qquad (j \in \mathbb{N}).$$

Then we have

$$\int_{(G_N)^c} \Big| \sum_{k=1}^{|\Gamma_n|} c_k D_k(x) \Big| \, dx \le \sum_{j=0}^{N-1} \sum_{\ell=0}^{m_j-1} |G_j|^{-1} \int_{(G_N)^c} \xi_{G_j}(x) \Big| \sum_{k=1}^{|\Gamma_n|} k_{j,\ell} c_k \chi_k(x) \Big| \, dx.$$

Introducing $h_{j,\ell}(x) = \operatorname{sgn}\Big(\sum_{k=1}^{|\Gamma_n|} k_{j,\ell} c_k \chi_k(x) \Big)$, this becomes

$$\begin{aligned} \int_{(G_N)^c} \Big| \sum_{k=1}^{|\Gamma_n|} c_k D_k(x) \Big| \, dx &\le \sum_{j=0}^{N-1} \sum_{\ell=0}^{m_j-1} |G_j|^{-1} \sum_{k=1}^{|\Gamma_n|} k_{j,\ell} c_k \int_G \xi_{G_j}(x) h_{j,\ell}(x) \chi_k(x) dx \\ &= \sum_{j=0}^{N-1} \sum_{\ell=0}^{m_j-1} |G_j|^{-1} \sum_{k=1}^{|\Gamma_n|} k_{j,\ell} c_k \overline{(\xi_{G_j} \overline{h_{j,\ell}})^\wedge(k)} \end{aligned}$$

We will apply Hölder's inequality followed by Hausdorff- Young's and in the final step the boundedness of the Vilenkin group to obtain

$$\begin{aligned}\int_{(G_N)^c}\Big|\sum_{k=1}^{|\Gamma_n|}c_kD_k(x)\Big|\,dx &\le C\sum_{j=0}^{N-1}\sum_{\ell=0}^{m_j-1}|G_j|^{-1}\Big(\sum_{k=1}^{|\Gamma_n|}|c_k|^q\Big)^{1/q}\\ &\quad\times\Big(\sum_{k=1}^{|\Gamma_n|}|(\xi_{G_j}h_{j,\ell})^\wedge(k)|^{q'}\Big)^{1/q'}\\ &\le C\sum_{j=0}^{N-1}\sum_{\ell=0}^{m_j-1}|G_j|^{-1}\Big(\sum_{k=1}^{|\Gamma_n|}|c_k|^q\Big)^{1/q}\|\xi_{G_j}h_j\|_q\\ &\le C\sum_{j=0}^{N-1}\sum_{\ell=0}^{m_j-1}|G_j|^{\frac{1}{q}-1}\Big(\sum_{k=1}^{|\Gamma_n|}|c_k|^q\Big)^{1/q}\\ &\le C|G_N|^{\frac{1}{q}-1}\Big(\sum_{k=1}^{|\Gamma_n|}|c_k|^q\Big)^{1/q}.\end{aligned}$$

□

Our theorem about the generalized Marcinkiewicz condition [7] reads as follows.

Theorem 9. *Let G be a compact multiplicative Vilenkin group. Suppose that $1<q\le 2$ and $p>\dfrac{q}{2q-1}$. If φ is bounded and satisfies*

$$\sum_{j\in\Gamma_{k+1}\setminus\Gamma_k}|\varphi(j+1)-\varphi(j)|^q\le C\,|\Gamma_k|^{1-q}\,,$$

then T_φ is bounded on $H^p(G)$.

Remark. We note that besides the trigonometric and the Vilenkin systems the Marcinkiewicz condition have been studied with respect to some other systems as well. Here we only mention a recent result by Weisz [14] in which the Ciesielski system is considered.

Proof. We will show the above Marcinkiewicz condition implies the kernel satisfies the Kitada-Tateoka condition from Corollary 6 to provide boundedness on $H^p(G)$. Recall that this condition for G compact is

$$\sum_{n=0}^{k}|G_n|^{1-p}\Big(\int_{G_n\setminus G_{n+1}}|\Delta_k\mathbb{k}(y)|\,dy\Big)^p\le C|G_k|^{1-p}.$$

We begin with the left-hand side:

$$\begin{aligned}I_1&=\sum_{n=0}^{k}|G_n|^{1-p}\Big(\int_{G_n\setminus G_{n+1}}|\Delta_k\mathbb{k}(y)|\,dy\Big)^p\\ &=\sum_{n=0}^{k}|G_n|^{1-p}\Big(\int_{G_n\setminus G_{n+1}}\Big|\sum_{m=|\Gamma_k|}^{|\Gamma_{k+1}|}\varphi(m)\chi_m(y)\Big|\,dy\Big)^p.\end{aligned}$$

For the inner sum, we use summation by parts to obtain:

$$\begin{aligned}\Big|\sum_{m=|\Gamma_k|}^{|\Gamma_{k+1}|}\varphi(m)\chi_m\Big|&\le\big|\varphi(|\Gamma_k|)D_{|\Gamma_k|}\big|+\big|\varphi\left(|\Gamma_{k+1}|\right)D_{|\Gamma_{k+1}|}\big|\\ &\quad+\Big|\sum_{m=|\Gamma_k|}^{|\Gamma_{k+1}|-1}(\varphi(m+1)-\varphi(m))D_m\Big|.\end{aligned}$$

Consequently,

$$
\begin{aligned}
I_1 \leq & \sum_{n=0}^{k} |G_n|^{1-p}\Big(\int_{G_n\backslash G_{n+1}} \left|\varphi\left(|\Gamma_k|\right) D_{|\Gamma_k|}(y)\right| + \left|\varphi\left(|\Gamma_{k+1}|\right) D_{|\Gamma_{k+1}|}(y)\right| dy\Big)^p \\
& + \sum_{n=0}^{k} |G_n|^{1-p}\Big(\int_{G_n\backslash G_{n+1}} \Big|\sum_{m=|\Gamma_k|}^{|\Gamma_{k+1}|-1} (\varphi(m+1)-\varphi(m))D_m(y)\Big| dy\Big)^p \\
=& I_{11} + I_{12}.
\end{aligned}
$$

For I_{11}, we are integrating over $G_n\backslash G_{n+1}$ which is contained in the complement of the support of $D_{|\Gamma_k|}$ and $D_{|\Gamma_{k+1}|}$ for $n<k$. So in this case the integral is zero. For $n=k$, we have

$$
\begin{aligned}
I_{11} &= |G_k|^{1-p}\Big(\int_{G_k\backslash G_{k+1}} \left|\varphi\left(|\Gamma_k|\right) D_{|\Gamma_k|}(y)\right| + \left|\varphi\left(|\Gamma_{k+1}|\right) D_{|\Gamma_{k+1}|}(y)\right| dy\Big)^p \\
&\leq |G_k|^{1-p}\left(B\,|\Gamma_k|\,|G_k| + 0\right)^p \\
&= B^p\,|G_k|^{1-p},
\end{aligned}
$$

where B is an upper bound for $|\varphi|$. This is the desired estimate for I_{11}. For I_{12} we apply the Sidon type inequality in Lemma 8:

$$
\begin{aligned}
I_{12} &= \sum_{n=0}^{k} |G_n|^{1-p}\Big(\int_{G_n\backslash G_{n+1}} \Big|\sum_{m=|\Gamma_k|}^{|\Gamma_{k+1}|-1} (\varphi(m+1)-\varphi(m))D_m(y)\Big| dy\Big)^p \\
&\leq C\sum_{n=0}^{k} |G_n|^{1-p}\Big(|G_n|^{\frac{1}{q}-1}\Big(\sum_{m=|\Gamma_k|}^{|\Gamma_{k+1}|-1} |\varphi(m+1)-\varphi(m)|^q\Big)^{1/q}\Big)^p \\
&\leq C\sum_{n=0}^{k} |G_n|^{1-p}\left(|G_n|^{\frac{1}{q}-1}\,|G_k|^{1-\frac{1}{q}}\right)^p \\
&\leq C\,|G_k|^{(1-\frac{1}{q})p}\sum_{n=0}^{k} |G_n|^{1-2p+\frac{p}{q}} \\
&\leq C\,|G_k|^{(1-\frac{1}{q})p}\,|G_k|^{1-2p+\frac{p}{q}} \quad \text{as } 1-2p+\frac{p}{q}>0 \\
&= C\,|G_k|^{1-p},
\end{aligned}
$$

the desired estimate for I_{12}. This completes the proof. □

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