Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
20 (2004), 153–163
www.emis.de/journals
ISSN 1786-0091

MODIFIED DYADIC DERIVATIVES AND INTEGRALS OF FRACTIONAL ORDER ON R_+

B.I. GOLUBOV

Dedicated to the 60th birthday of Professor W.R. Wade

Abstract. We give a brief review of the known results on pointwise and strong dyadic differentiation and integration of real functions. In section 3 some new results on modified dyadic fractional differentiation and integration are formulated.

Introduction

Following the concept of J.E. Gibbs [1] P.L. Butzer and H.J. Wagner [2] defined dyadic strong derivative D. After that they introduced dyadic pointwise derivative d and dyadic strong integral I (see $[3]-[5]$). Their definitions concerns to functions defined on dyadic group G or dyadic field K. Dyadic group G and dyadic field K are isomorphic to modified segment $[0,1]^*$ and modified positive half-line $R_+^* = [0,+\infty)^*$ respectively. The characters of dyadic group G and dyadic field K are Walsh-Paley functions $w_n(\cdot)$, $n \in Z_+ = \{0,\ 1,\ 2,\dots\}$ and generalized Walsh functions $\psi_y(\cdot)$, $y \in R_+$ respectively. P.L. Butzer and H.J. Wagner proved the equalities $D\,w_n = n\,w_n$ and $d\,w_n(x) = n\,w_n(x)$ for $n \in Z_+$, $x \in G$ and $d\,\psi_y(x) = |y|\psi_y(x)$ for $x, y \in K$.

C.W. Onneweer [6] introduced modified pointwise and strong dyadic derivatives for functions defined on dyadic group G or dyadic field K. He proved that the characters of dyadic group G or dyadic field K are differentiable in his sense and they are eigenfunctions of modified differential operator δ. For example, he proved the equalities

$$\delta(w_0)\,(x) \equiv 0, \quad \delta(w_n)\,(x) = 2^k w_n(x), \quad 2^k \le n < 2^{k+1}, \quad k \in Z_+, \quad x \in D.$$

In another article [7] C.W. Onneweer introduced modified fractional differentiation and integration on compact Vilenkin groups G_p of order $p \ge 2$ and proved fundamental theorem of dyadic calculus.

In this paper we give a brief outline of known results concerning dyadic derivatives and integrals.

We also define modified dyadic strong and pointwise integrals and derivatives of fractional order on R_+ and formulate some results concerning their properties.

2000 *Mathematics Subject Classification.* 26A33, 44A35, 44A15.

Key words and phrases. Walsh series, Walsh transformation, dyadic fractional differentation and integration.

This work was supported by the Russian Foundation for Basic Research, Grant 02-01-00428.

1. Notations and definitions

For a number $x \in R_+ \equiv [0, +\infty)$ we consider dyadic expansion

$$x = \sum_{n=-\infty}^{+\infty} 2^{-n-1} x_n,$$

where x_n equals to 0 or 1. Note that $x_n = 0$ for $n \le n(x)$, where $n(x) \in Z = \{0, \pm 1, \pm 2, \dots\}$. If x is dyadic rational, then we take its finite expansion, i.e. $x_n = 0$ for $n \ge n_0(x) > -\infty$. We define dyadic sum of two numbers $x, y \in R_+$ by the operation $\oplus$ as follows: $x \oplus y = z$, where $z_n = x_n + y_n \pmod 2$ for all $n \in Z$. Let us put $t(x, y) = \sum\limits_{n=-\infty}^{+\infty} x_n y_{-n-1}$ and define the generalized Walsh functions

$$\psi(x, y) \equiv \psi_y(x) = (-1)^{t(x, y)} \quad \text{for} \quad (x, y) \in R_+ \times R_+.$$

They were introduced by N.J. Fine [8]. It is evident that $\psi(x, y) = \psi(y, x)$, $\psi(x, y) = \pm 1$ for $x, y \in R_+$. The functions $w_n(x) \equiv \psi(x, n)$, $n \in Z_+$, are called the Walsh-Paley functions. They are 1-periodic on R_+. It is evident that $w_0(x) \equiv 1$ on R_+. The system $\{w_n(x)\}_{n=0}^{+\infty}$ is orthonormal on $[0, 1)$, i.e.

$$\int_0^1 w_m(x) w_m(x)\, dx = \delta_{m,n},$$

where $\delta_{m,n}$ is Kronecker symbol, i.e. $\delta_{m,n} = 0$ for $m \ne n$ and $\delta_{n,n} = 1$.

Let be given a function $f \in L[0, 1)$. We denote by $\sum\limits_{n=0}^{+\infty} \hat{f}(n) w_n(x)$ its Fourier series with respect to the Walsh-Paley system, where $\hat{f}(n) = \int_0^1 f(x) w_n(x)\, dx$, $n \in Z_+$, are Walsh-Fourier coefficients of the function f.

For the function $f \in L(R_+)$ N.J. Fine [8] introduced its Walsh transform by the equality

$$F[f]\,(x) \equiv \tilde{f}(x) = \int\limits_{R_+} \psi(x, y)\, f(y)\, dy.$$

If $f \in L^p(R_+)$, $1 < p \le 2$, then its Walsh transform is defined as the limit as $n \to +\infty$ of the sequence $\int\limits_0^{2^n} f(y)\, \psi(x, y)\, dy$ in the norm of the space $L^q(R_+)$, where $1/p + 1/q = 1$.

For $f \in L(R_+)$, $g \in L^p(R_+)$, $1 \le p \le +\infty$, we set

$$(f * g)(x) \int\limits_{R_+} = f(x \oplus y) g(y)\, dy, x \in R_+,$$

i.e. $f * g$ is dyadic convolution of f and g. Let us note that $f * g \in L^p(R_+)$, $(f * \tilde{g)} = \tilde{f}\tilde{g}$.

The function $f \in L(R_+)$ is called $W-$continuous at the point $x \in R_+$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x \oplus y) - f(x)| < \varepsilon$ for $0 < y < \delta$ (see [9], Chapter 1).

Let us note that the Wash-Fourier transform $\tilde{f}$ of every function $f \in L(R_+)$ is W-continuous on R_+ (see [9], Theorem 6.1.5).

We call the point $x \in R_+$ dyadic Lebesgue point of local integrable function f, if f is defined at the point x and

$$\lim_{n \to +\infty} 2^n \int_0^{2^{-n}} |f(x \oplus t) - f(x)|\, dt = 0.$$

Almost all points of local integrable function are its dyadic Lebesgue points. If a function f is $W-$continuous at the point $x \in R_+$, then x is its dyadic Lebesgue point. (There is also a concept of Walsh-Lebesgue point of an integrable on $[0,1)$ function, see [26], [27]).

Let us define the generalized Walsh-Dirichlet integral of the function $f \in L(R_+)$ by the equality

$$S_y(f)\,(x) = \int_0^y \tilde{f}(t)\psi(x,t)\,dt.$$

Theorem. *If $x \in R_+$ is dyadic Lebesgue point of the function $f \in L(R_+)$, then*

$$\lim_{n\to+\infty} S_{2^n}(f)(x) = f(x).$$

The statement of this theorem was proved at the page 430 in [10] for the points of $W-$continuity of the function f. But the proof is valid also for dyadic Lebesgue points.

It follows from this theorem that if $f, \tilde{f} \in L\,(R_+)$, then $f(x) = \int\limits_{R_+} \psi(x,y)\,\tilde{f}(y)\,dy$ almost everywhere (a.e.) on R_+.

Let $\Delta = \{\Delta_n^k\}$ denote the set of all dyadic intervals $\Delta_n^k \equiv [k2^{-n},\,(k+1)2^{-n})$, $k \in Z_+$, $n \in Z$. Let us introduce dyadic maximal function

$$M_d(f)(x) = \sup_{x\in I\in\Delta}\left|\frac{1}{|I|}\int_I f(t)\,dt\right|, \quad x \in R_+,$$

and dyadic Hardy space

$$H(R_+) = \{f \in L(R_+):\; M_d(f) \in L(R_+)\}.$$

The norm on $H(R_+)$ is $\|f\|_{H(R_+)} = \|M_d(f)\|_{L(R_+)}$.

By similar way dyadic Hardy space $H([0,1))$ is defined.

Below $C_W(R_+)$ is the space of uniformly $W-$continuous functions on R_+. The norm on the $C_W(R_+)$ is $\|f\|_{C_W(R_+)} = \sup\limits_{x\in R_+}|f(x)|$. The symbol $C_W[0,1)$ will denote the space of uniformly $W-$continuous functions on $[0,1)$ with the norm $\|f\|_{C_W[0,1)} = \sup\limits_{x\in[0,1)}|f(x)|$. For the sake of convenience we shall consider the spaces $C_W[0,1)$ and $C_W(R_+)$ as the spaces $L^p[0,1)$ or $L^p(R_+)$ respectively for $p = +\infty$.

2. The known concepts of dyadic derivatives and integrals

Dyadic derivatives. P.L. Butzer and H.J. Wagner [4] defined dyadic pointwise derivative as follows.

Definition 2.1. Let be given the function $f \in L[0,1)$ and a point $x \in [0,1)$. If there exists finite limit

$$d^{(1)}f(x) = \lim_{n\to+\infty}\sum_{m=0}^{n} 2^{m-1}\left[f(x) - f(x \oplus 2^{-m-1})\right],$$

then $d^{(1)}f(x)$ is called dyadic derivative of the function f at the point x. The dyadic derivatives of higher order are defined by recurrence formulae

$$d^{(m)}f(x) = d^{(1)}(d^{(m-1)}f)\,(x), \quad m = 2,3,\ldots$$

P.L. Butzer and H.J. Wagner proved that each Walsh-Paley function has dyadic derivative at each point $x \in [0,1)$ and $d^{(1)}w_n(x) = n\,w_n(x)$ for $n \in Z_+$.

The notion of strong dyadic L^p-derivative was introduced by P.L. Butzer and H.J. Wagner [2] by the following way.

Definition 2.2. If for the function $f \in L^p[0,1)$, $1 \le p \le +\infty$, the limit

$$D^{(1)}(f)_{L^p} \equiv (L^p) - \lim_{n\to+\infty} \sum_{m=0}^{n} 2^{m-1}[f(\cdot) - f(\cdot + 2^{-m-1})]$$

exists in the norm of the space $L^p[0,1)$, then it is called $L^p[0,1)-$ derivative of the function f. The strong dyadic L^p-derivatives of higher order are defined by recurrence formula $D^{(m)}(f)_{L^p} = D^{(1)}((D^{(m-1)f})_{L^p})_{L^p}$, $m = 2, 3, \ldots$

It is proved in [2] that every Walsh function has strong dyadic $L^p[0,1)-$derivative of arbitrary order $r \in N$ for each $1 \le p \le +\infty$ and $D^{(r)}(w_n)_{L^p} = n^r w_n$ for $n \in Z_+$.

P.L. Butzer and H.J. Wagner [2] proved the following

Theorem 2.1. *If a function $f \in L^p[0,1)$, $1 \le p \le +\infty$, has strong dyadic $L^p[0,1)$-derivative $D^{(r)}(f)_{L^p} = g$, then $\hat{g}(n) = n^r \hat{f}(n)$, $n \in Z_+$, where $\hat{f}(n)$ are Walsh-Fourier coefficients of the function f.*

C.W. Onneweer [11] generalized the concepts of pointwise dyadic derivative and strong dyadic $L^p[0,1)$-derivative to functions defined on Vilenkin groups.

For functions f defined on R_+ the natural analogue of pointwise dyadic derivative $d^{(1)}f(x)$ is

$$df(x) = \lim_{n\to+\infty} \sum_{m=-n}^{n} 2^{m-1}(f(x) - f(x \oplus 2^{-m-1})).$$

(see [3]). P.L. Butzer and H.J. Wagner [3] proved that the generalized Walsh functions have dyadic derivative at each point. More precisely $d\psi_y(x) = y\psi_y(x)$, $x \in R_+$.

For the functions $f \in L^p[0,1)$, $1 \le p \le +\infty$ the strong dyadic $L^p(R_+)$-derivative is defined as follows:

$$D\,(f)_{L^p(R_+)} = \lim_{n\to\infty} \sum_{m=-n}^{n} 2^{m-1}[f(\cdot) - f(\cdot \oplus 2^{-m-1})],$$

where the limit is taken in the norm of the space $L^p(R_+)$ (see [5]). The notion of $L^p(R_+)$-derivative $D^{(r)}(f)_{L^p(R_+)}$ of higher order $r = 2, 3, \ldots$ is defined by recurrence formula.

It is known that if $f \in L^p(R_+)$, $p = 1$ or 2, and $D\,(f)_{L^p(R_+)}$ exists, then

$D\,(\tilde{f})_{L^p(R_+)}(x) = x\,\tilde{f}(x)$. (For $p = 1$ it was proved by P.L. Butzer and H.J. Wagner [3]; for $p = 2$ see J. Pál [12]). C.W. Onneweer [6] introduced *modified* pointwise and strong dyadic derivatives for functions defined on dyadic group G or dyadic field K. (The characters of dyadic field K are generalized Walsh functions $\psi_y(\cdot)$, $y \in R_+$, and the characters of the group G are Walsh-Paley functions w_n, $n \in Z_+$). He proved that the characters of dyadic group G or dyadic field K are differentiable in his sense at each point and they are eigenfunctions of modified differential operator δ. For example, he proved the equalities

$$\delta(w_0)\,(y) \equiv 0, \quad \delta(w_n)\,(y) = 2^k w_n(y), \quad 2^k \le n < 2^{k+1}, \quad k \in Z_+, \quad y \in D.$$

In another article [7] C.W. Onneweer introduced modified fractional differentiation and integration on compact groups of order $p \ge 2$ and proved fundamental theorem of dyadic calculus.

Dyadic integrals. The *dyadic integral* for functions defined on the interval $[0,1)$ was introduced by P.L. Butzer and H.J. Wagner [2] as follows. Let us set

$$W_r(x) = 1 + \sum_{n=1}^{+\infty} \frac{w_n(x)}{n^r}, \quad r \in N.$$

It is evident that $W_r \in L[0,1)$, $r \in N$. If $f \in L^p[0,1)$, $1 \le p \le +\infty$, then there exists dyadic convolution

$$I_r(f) = (f * W_r)(x) \equiv \int_0^1 f(y)\, W_r(x \oplus y)\, dy, \quad r \in N \tag{$*$}$$

and $I_r(f) \in L^p[0,1)$. The function $I_r(f)$ is called *dyadic strong integral of order r of the function f in the space* $L^p[0,\,1)$.

It follows from $(*)$ that for $f \in L^p[0,1)$, $1 \le p \le +\infty$ its dyadic integral $I_r(f)$ has Walsh-Paley series of the form

$$\hat{f}(0) + \sum_{n=0}^{+\infty} \frac{\hat{f}(n)}{n^r} w_n.$$

P.L. Butzer and H.J. Wagner [2] proved the following fundamental theorem of dyadic calculus.

Theorem 2.2. *Let* $f \in L^p[0,1)$, $1 \le p \le +\infty$ *and* $\hat{f}(0) = 0$.
a) *If there exists* $L^p[0,1)$*-derivative* $D^{(r)}(f)_{L^p}$ *of some order* $r \in N$, *then* $I_r(D^{(r)}(f)_{L^p}) = f$.
b) *One has* $D^{(r)}(I_r(f))_{L^p} = f$ *for all* $r \in N$.

J. Pál and P. Simon [13] generalized the concept of strong dyadic $L^p[0,\,1)$-integral to functions defined on Vilenkin groups. They proved a generalization of the Theorems 2.1 (for $p = 1$) and 2.2, using the concept of strong dyadic $L^p[0,\,1)$-derivative for functions defined on Vilenkin groups due to C.W. Onneweer.

F. Schipp [14] proved that dyadic strong integral has pointwise dyadic derivative a.e. More precisely the following theorem is valid.

Theorem 2.3. *If* $f \in L[0,\,1)$, *then* $d^{(1)}(I_1(f))\,(x) = f(x)$ *a.e. on* $[0,\,1)$, *where* $I_1(f)$ *is dyadic strong integral of first order of the function f in the space* $L[0,1)$.

For the functions $f \in L^p(R_+)$, $1 \le p \le +\infty$, the strong dyadic integral was defined by H. J. Wagner [5] as follows. For $n \in Z_+$ we set

$$W_n(x) = \lim_{k \to +\infty} \int_{2^{-n}}^{2^k} \frac{1}{t} \psi_x(t)\, dt, \quad x \in R_+.$$

It has been proved in [5] that this limit exists a.e. on R_+ and also in $L(R_+)$-metric. Therefore there exists dyadic convolution

$$(f * W_n)(x) = \int_{R_+} f(t) W_n(x \oplus t)\, dt, \quad n \in Z_+, \tag{$**$}$$

and $f * W_n \in L^p(R_+)$, if $f \in L^p(R_+)$, $1 \le p \le +\infty$.

Definition 2.3. If for a function $f \in L^p(R_+)$, $1 \le p \le +\infty$, the sequence $(**)$ converges in $L^p(R_+)$-metric to a function $g \in L^p(R_+)$ as $n \to +\infty$, then $g \equiv I(f)$ is called strong dyadic integral of the function f in the space $L^p(R_+)$ or shortly $L^p(R_+)$-integral of the function f.

The notion of $L^p(R_+)$-integral $I_r(f)$ of higher order $r = 2, 3, \ldots$ is defined by recurrence formula.

The following results were proved by H.J. Wagner [5]:

Theorem 2.4. *For two functions* $f, g \in L(R_+)$ *the equality* $g = I(f)$ *holds if and only if* $\tilde{g}(0) = 0$ *and* $\tilde{g}(x) = \tilde{f}(x)/x$, $x > 0$, *where* $I(f)$ *is* $L(R_+)$*-integral of the function* f.

Theorem 2.5. *Let be given a function $f \in L(R_+)$.*
a) *If $L(R_+)$-integral $I(f)$ exists, then $D(I(f))_{L(R_+)} = f$.*
b) *If $L(R_+)$-derivative $D(f)_{L(R_+)}$ exists and $\tilde{f}(0) = 0$, then $I(D(f)_{L(R_+)}) = f$.*

J. Pál and F. Schipp [15] proved the following theorem.

Theorem 2.6. *If a function $L(R_+)$ has strong dyadic integral $g = I(f)$ in the space $L(R_+)$, then $I(f)$ has pointwise dyadic derivative a.e. on R_+ and $d(I(f))(x) = f(x)$ a.e. on R_+.*

3. Modified dyadic integral and derivative of fractional order on R_+

Strong and pointwise derivatives and integrals of fractional order on R_+. . In this subsection we formulate our results most of which are analogues of the results of C.W. Onneweer [7] concerning the functions defined on compact groups G_p of order $p = 2, 3, \dots$

For $x > 0$ we set $h(x) = 2^{-n}$, $2^n \le x < 2^{n+1}$, $n \in Z$. It is evident that $x^{-1} \le h(x) < 2\,x^{-1}$.

Lemma 3.1. *If $\alpha > 0$ and $n \in Z$, then for each $x > 0$ there exists finite limit*

$$W_n^\alpha(x) = \lim_{m\to+\infty} \int_{2^{-n}}^{2^m} (h(y))^\alpha\, \psi_x(y)\, dy.$$

More precisely, $W_n^\alpha(x) = -2^{(\alpha-1)\,n}$ for $2^{n-1} \le x < 2^n$,

$$W_n^\alpha(x) = -2^{(\alpha-1)\,n} + 2(1-2^{-\alpha}) \sum_{i=0}^{k} 2^{(n-i)\,(\alpha-1)}$$

for $2^{n-k-2} \le x < 2^{n-k-1}$, $k = 0, 1, \dots$ and $W_n^\alpha(x) = 0$ for $x \ge 2^n$.

We shall write $f(x) \approx g(x)$, $x \to a$, if $f(x) = O(g(x))$, $x \to a$, and $g(x) = O(f(x))$, $x \to a$. Then we have the following corollary from the lemma 3.1.

Corollary 3.1. *1) If $0 < \alpha < 1$, $n \in Z$, then $W_n^\alpha(x) \approx x^{\alpha-1}$, $x \to +0$; 2) $W_n^1(x) \approx \log_2(x^{-1})$, $x \to +0$; 3) if $\alpha > 1$, then $W_n^\alpha(x)$ is bounded on R_+; 4) $W_n^\alpha \in L(R_+)$ for all $\alpha > 0$, $n \in Z$.*

Definition 3.1. If $\alpha > 0$, $f, g \in L^p(R_+)$, and $\lim\limits_{n\to+\infty} \|f * W_n^\alpha - g\|_{L^p(R_+)} = 0$, then the function $g = J_\alpha(f)$ is called modified strong dyadic integral (MSDI) of order α of the function f in the space $L^p(R_+)$.

Theorem 3.1. *Let $f, g \in L(R_+)$ and $\alpha > 0$. Then the function g is MSDI of order α of the function f in the space $L(R_+)$, if and only if $\tilde{g}(0) = 0$ and $\tilde{g}(x) = \tilde{f}(x)\,(h(x))^\alpha$ for $x > 0$.*

Let us set for $\alpha > 0$, $n \in Z$:

$$\Lambda_n^\alpha(x) = \int_0^{2^n} (h(t))^{-\alpha} \psi(x,t)\, dt, \quad x \in R_+ .$$

Lemma 3.2. *For $\alpha > 0$, $n \in Z$ we have $\Lambda_n^\alpha \in L(R_+) \cap L^\infty(R_+)$.*

Definition 3.2. If $\alpha > 0$, $f, \varphi \in L^p(R_+)$, $1 \le p \le +\infty$, and

$$\lim_{n\to+\infty} \|f * \Lambda_n^\alpha - \varphi\|_{L^p(R_+)} = 0,$$

then the function $\varphi = D^\alpha(f)$ is called modified strong dyadic derivative (MSDD) of order α of the function f in the space $L^p(R_+)$.

Theorem 3.2. *Let $\alpha > 0$ and $f, \varphi \in L^p(R_+)$, $1 \le p \le 2$. Then the function φ is MSDD of order α of the function f in the space $L^p(R_+)$ if and only if*

$$\tilde{\varphi}(x) = \tilde{f}(x)\,(h(x))^{-\alpha}$$

a.e. on R_+.

This theorem is a corollary from the R_+-version of a theorem of C.W. Onneweer (see [22], Theorem 3).

Theorem 3.3. *Let $\alpha > 0$ and the function $f \in L(R_+)$ has MSDD $D^\alpha(f)$ of order α in the space $L(R_+)$. If $\tilde{f}(0) = 0$, then the equality $J_\alpha(D^\alpha(f)) = f$ holds.*

Theorem 3.4. *Let $\alpha > 0$ and the function $f \in L(R_+)$ has MSDI $J_\alpha(f)$ of order α in the space $L(R_+)$. Then the equality $D^\alpha(J_\alpha(f)) = f$ is valid.*

The Theorems 3.3 and 3.4 are R_+-version of fundamental theorem of dyadic calculus (see Theorem 2.2 above).

Theorem 3.5. *Let $\alpha > 0$, $\beta > 0$ and $f \in L(R_+)$. Then $D^\alpha(D^\beta(f)) = D^{\alpha+\beta}(f)$ (respectively $J^\alpha(J^\beta(f)) = J^{\alpha+\beta}(f)$), if the left side of this equality exists.*

Theorem 3.6. *The functions $a_{m,n}(x) = \psi(x, m2^{-n)})X_{[0,2^n)}(x)$, $m \in N$, $n \in Z$, for each $\alpha > 0$ are eigenfunctions of the operators J_α and D^a with eigenvalues $2^{-r\alpha}$ and $2^{r\,\alpha}$ respectively. Here X_E is indicator function of the set E and $r = r\,(m,n) \in Z$ is uniquely determined by the imbedding $[m2^{-n}, (m+1)2{-}n] \subset [2^r, 2^{r+1})$.*

Let us denote by $L_{J_\alpha}(R_+)$ or $L_{D^\alpha}(R_+)$ the natural domain of the operator J_α or D^α respectively, i.e. the set of all functions $f \in L(R_+)$ for which $J_\alpha(f)$ or $D^\alpha(f)$ respectively exists. It is evident that $L_{J_\alpha}(R_+)$ and $L_{D^\alpha}(R_+)$ are linear subspaces in $L(R_+)$.

It follows from the Theorem 3.6 that

$$J_\alpha(a_{1,\,n}) = 2^{n\,\alpha}a_{1,\,n}, \quad D^\alpha(a_{1,\,n}) = 2^{-n\,\alpha}a_{1,\,n}, \quad n \in Z, \quad \alpha > 0.$$

Therefore we have

Corollary 3.2. *The linear operators $J_\alpha\colon L_{J_\alpha}(R_+) \to L(R_+)$ and $D^\alpha\colon L_{D^\alpha}(R_+) \to L(R_+)$ are unbounded for each $\alpha > 0$.*

Let us define the pointwise dyadic derivative of fractional order. According to the lemma 3.2 we have $\Lambda_n^\alpha \in L^\infty(R_+) \cap L(R_+)$ for $\alpha > 0$, $n \in Z$. Therefore the dyadic convolution $(\Lambda_n^\alpha * f)\,(x)$ exists at each point $x \in R_+$ for all $\alpha > 0$, $n \in Z$, if $f \in L(R_+)$ or $f \in L^\infty(R_+)$. Taking into account this fact we may to introduce the following definition.

Definition 3.3. Let $\alpha > 0$, $x \in R_+$ and $f \in L(R_+)$ or $f \in L^\infty(R_+)$. If there exists finite limit $d^\alpha(f)\,(x) \equiv \lim\limits_{n\to+\infty}(\Lambda_n^\alpha * f)(x)$, then we shall say that the function f has the modified dyadic derivative (MDD) $d^\alpha(f)(x)$ of order α at the point x.

Theorem 3.7. *For each $\alpha > 0$ and fixed $y \in R_+$ the Walsh generalized function $\psi_y(\cdot)$ has MDD of order α at each point $x \in R_+$. More precisely, $d^\alpha(\psi_0)(x) \equiv 0$ on R_+ and $d^\alpha(\psi_y)(x) = (h(y))^{-\alpha}\psi_y(x)$ for $x \in R_+$, $y > 0$.*

For the case $\alpha = 1$ these results were published in [21].

Theorem 3.8. *If $\alpha > 0$ and the function $f \in L(R_+)$ is such that $(h(x))^{-\alpha}\tilde{f}(x) \in L(R_+)$, then at each point $x \in R_+$ it has MDD of order α equal to $\int_0^{+\infty}(h(y))^{-\alpha}\tilde{f}(y)\psi(x,\,y)\,dy$.*

Theorem 8 is an analogue of the following theorem of P.L. Butzer and H.J. Wagner [4].

Theorem 3.9. *Under the assumption* $\sum\limits_{n=0}^{\infty} n|\alpha_n| < +\infty$ *the series* $\sum\limits_{n=0}^{\infty} a_n w_n(x)$ *is absolutely and uniformly convergent on* $[0,\,1)$ *to a function* f*, which has dyadic derivative* $d^{(1)}f(x)$ *for all* $x \in [0,\,1)$ *and* $d^{(1)}f(x) = \sum\limits_{n=0}^{+\infty} n\,a_n w_n(x)$.

Pointwise and strong dyadic term by term differentiation of Walsh series was investigated by W.R. Wade [23], V.A. Skvorčov and W.R. Wade [24], C.H. Powell and W.R. Wade [25].

Let us define the pointwise dyadic integral of fractional order. According to the corollary 3.1 we have $W_n^\alpha \in L(R_+)$. Therefore the dyadic convolution $(W_n^\alpha * f)(x)$ exists at each point $x \in R_+$ for all $\alpha > 0$, $n \in Z$, if $f \in L^\infty(R_+)$. Taking into account we may to introduce the following definition.

Definition 3.4. If $\alpha > 0$, $x \in R_+$ and for a function $f \in L^\infty(R_+)$ there exists finite limit $j_\alpha(f)(x) \equiv \lim\limits_{n\to+\infty}(f * W_n^\alpha)(x)$, then we say that the function f has modified dyadic integral (MDI) of order α at the point x equal to $j_\alpha(f)(x)$.

Theorem 3.10. *For each* $\alpha > 0$ *and fixed* $y \in R_+$ *the generalized Walsh function* $\psi_y(\cdot)$ *has MDI of order* α *at each point* $x \in R_+$*. More precisely,* $j_\alpha(\psi_0)(x) \equiv 0$ *on* R_+ *and* $j_\alpha(\psi_y)(x) = (h(y))^\alpha \psi_y(x)$ *for* $x \in R_+$, $y > 0$.

Definition 3.5. Let $\alpha > 0$ and $x \in R_+$. If for the function $f \in L(R_+)$ there exists finite limit

$$d^{(\alpha)}(f)(x) \equiv \lim_{n\to+\infty} \int_0^{2^n} (h(y))^{-\alpha} \tilde{f}(y)\psi(x,y)\,dy,$$

then we shall call it dyadic α-derivative of the function f at the point x.

If $f \in L(R_+)$ and $(h)^{-\alpha}\tilde{f} \in L(R_+)$, then dyadic α-derivative of the function f exists at every point $x \in R_+$ and $d^{(\alpha)}(f)(x) = \int_{R_+}(h(y))^{-\alpha}\tilde{f}(y)\psi(x,y)\,dy$.

Definition 3.6. Let be given $\alpha > 0$, $x \in R_+$ and $f \in L(R_+)$. If there exists finite limit

$$j_{(\alpha)}(f)(x) \quad \equiv \quad \lim_{n\to+\infty} \int_{2^{-n}}^{2^n} (h(y))^{\alpha} \tilde{f}(y)\psi(x,y)\,dy,$$

then we shall call it dyadic α-integral of the function f at the point x.

If $f \in L(R_+)$ and $(h)^{\alpha}\tilde{f} \in L(R_+)$, then dyadic $\alpha-$integral of the function f exists at every point $x \in R_+$ and $j_{(\alpha)}(f)(x) = \int_{R_+}(h(y))^{\alpha}\tilde{f}(y)\psi(x,y)\,dy$.

The following theorem may be considered as a dyadic analogue of the classical theorem of Lebesgue on pointwise differentiation of Lebesgue integral.

Theorem 3.11. *Let* $\alpha > 0$ *and the function* $f \in L(R_+)$ *has MSDI* $J_\alpha(f)$ *of order* α *in the space* $L(R_+)$*. Then at each Lebesgue point* $x \in R_+$ *of the function* f*, hence a.e. on* R_+*, the equality* $d^{(\alpha)}(J_\alpha(f))(x) = f(x)$ *holds.*

Theorem 3.12. *Let* $\alpha > 0$ *and the function* $f \in L(R_+)$ *has MSDD* $D^{(\alpha)}(f)$ *of order* α *in the space* $L(R_+)$*. Then at each Lebesgue point* $x \in R_+$ *of the function* f*, hence a.e. on* R_+*, the equality* $j_{(\alpha)}(D^{(\alpha)}(f))(x) = f(x)$ *holds.*

Dyadic integral in dyadic Hardy space. The following theorem of Hardy [16] is well known.

Theorem 3.13. *If the function* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *belongs to the Hardy space* $H(|z| < 1)$ *on the unit disc* $|z| < 1$ *of the complex plane* C *and* $f(e^{it})$ *is its* *boundary function on the unit circle* $|z| = 1$*, then*

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \frac{1}{2} \int_0^{2\pi} |f(e^{it})| \, dt.$$

An analogue of this theorem has been proved by E. Hille and J.D. Tamarkin [17].

Theorem 3.14. *If the function* $f(z)$ *belongs to the Hardy space* $H(R_+^2)$ *on the upper half-plane* $R_+^2 = \{z \in C : \operatorname{Im} z > 0\}$ *and* $\hat{f}(x)$ *is Fourier transform of its boundary function* $f(x)$ *on real axis, then the following inequality holds*

$$\int_{R_+} \frac{|\hat{f}(x)|}{x} \, dx \leq \frac{1}{2} \int_{-\infty}^{+\infty} |f(x)| \, dx.$$

An extension of the Theorem 3.14 on Hardy space $H^p(R)$, $0 < p \leq 1$, is also known.

Theorem 3.15. *If* $f \in H^p(R)$*, then*

$$\int_{R_+} |\hat{f}(x)|^p x^{p-2} \, dx \leq C_p \|f\|_{H^p(R)}^p.$$

(See [18]*, p.342).*

Problem. *What are the least constants in right-hand sides of the inequalities of the Theorems 3.13–3.15?*

N.R. Ladhawala [19] proved a dyadic analogue of the Theorem 3.13 in the following form.

Theorem 3.16. *If the function* f *belongs to dyadic Hardy space* $H([0,1))$*, then*

$$\sum_{n=1}^{+\infty} \frac{|\hat{f}(n)|}{n} \leq 12\sqrt{2} \|f\|_H,$$

where $\hat{f}(n)$ *are Walsh–Fourier coefficients of the function* f*.*

A dyadic analogue of the Theorem 3.14 was proved in [20]:

Theorem 3.17. *If* $f \in H(R_+)$*, then the following inequality holds*

$$\int_{R_+} \frac{|\tilde{f}(x)|}{x} \, dx \leq 50\sqrt{2} \|f\|_{H(R_+)}.$$

Problem. *1) What is the least constant in right-hand side of the former inequality? 2) To extend this inequality on dyadic Hardy space* $H^p(R_+)$*,* $0 < p < 1$*, i.e. to prove dyadic analogue of the Theorem 3.15.*

Definition 3.7. Let us define the functions $(f * W_n^\alpha)^*(x)$, $n \in Z_+$, by the equality

$$\begin{aligned}(f * W_n^\alpha)^*(x) &\equiv \int_{2^{-n} \leq t \leq 2^n} (f * W_n^\alpha)\tilde{}(t) \psi_x(t) \, dt \\ &= \int_{2^{-n} \leq t \leq 2^n} \tilde{f}(t) (h(t))^\alpha \psi_x(t) \, dt, \quad x \in R_+,\end{aligned}$$

where $f \in L(R_+)$. If there exists the limit

$$J_\alpha^*(f)(x) \equiv \lim_{n \to +\infty} (f * W_n^\alpha)^*(x),$$

which is uniform on R_+, then we say that the function f has uniform modified dyadic integral (UMDI) of order α on R_+.

As a corollary from the Theorem 3.17 we obtain:

Theorem 3.18. *Each function $f \in H(R_+)$ has UMDI of first order on R_+. More precisely, the operator $J_1^* : H(R_+) \to C_w(R_+)$ is bounded and*

$$\|J_1^*(f)\|_{C_W(R_+)} \leq 100\sqrt{2}\|f\|_{H(R_+)}.$$

In [21] the following theorem is proved.

Theorem 3.19. *The functions $\psi(x, m2^{-n})\mathrm{X}_{[0,\,2^n)}(x)$, $m \in N$, $n \in Z$, belong to the space $H(R_+)$ and their linear hull L is dense in this space.*

If a function $f \in H(R_+)$ has modified dyadic strong integral $J_1(f)$ in the space $L(R_+)$, then $J_1(f)(x) = J_1^*(f)(x)$ a.e. on R_+. But the functions

$$\psi(x, m2^{-n})\mathrm{X}_{[0,\,2^n)}(x), \qquad m \in N, \quad n \in Z,$$

have modified strong dyadic integral of first order in the space $L(R_+)$ (see Theorem 3.6 above). Therefore by setting $\tilde{J}_1(f) \equiv J_1(f)$ we can deduce from the Theorems 3.1, 3.17 and 3.19 the following result.

Theorem 3.20. *The operator $\tilde{J}_1 \,:\, L \to L(R_+)$ is bounded and his operator norm does not exceed $100\sqrt{2}$. Therefore it can be extended continuously on the space $H(R_+)$ without changing its operator norm.*

This theorem may be considered as a dyadic analogue of the Theorem 3.14.

Acknowledgement

I would like to express my gratitude to the Organizing Committee of the International Conference "Dyadic Analysis with Applications and Generalizations" (Balatonszemes, June 11–13, 2003) for financial support during my staying at the conference.

References

[1] G.E. Gibbs. Walsh spectrometry, a form of spectral analysis well suited to binary digital computation, Nat. Phys. Lab., Teddington, UK, 24 p., 1967.
[2] P.L. Butzer, H.J. Wagner. Walsh series and the concept of a derivative Applicable Analysis **3**, No. 1 (1973), p. 29-46.
[3] P. L. Butzer, H. J. Wagner. A calculus for Walsh functions defined on R_+, Proc. Symp. Naval Res. Laboratory, Washington, D. C., April 18 - 20, 1973, p. 75-81.
[4] P. L. Butzer, H. J. Wagner. On dyadic analysis based on pointwise dyadic derivative, Anal. Math., **1** (1975), p. 171-196.
[5] H. J. Wagner. On dyadic calculus for functions defined on R_+, In: "Theory and applications of Walsh functions", Pros. Symp., Hatfield Polytechnic (1975), p.101-129.
[6] C.W. Onneweer. On the definition of dyadic differentiation, Applicable Anal., **9** (1979), p. 267 -278.
[7] C.W. Onneweer. Fractional differentiation on the group of integers of a p-adic or p-series field, Anal. Math., **3** (1977), p. 119-130.
[8] N. J. Fine. The generalized Walsh functions, Trans. Amer. Math. Soc., **69** (1950), p. 66-77.
[9] B. Golubov, A. Efimov, V. Skvortsov. Walsh series and transforms. Theory and applications. Dordrecht- Boston -London, Kluwer Academic Publishers, 1991.
[10] F. Schipp, W. R. Wade, P. Simon, J. Pál . Walsh series. An introduction to dyadic harmonic analysis. Budapest: Akademiai Kiado, 1990.
[11] C.W. Onneweer. Differentiability for Rademacher series on groups, Acta Sci. Math. Szeged, **39** (1977), p. 121-128.
[12] J. Pál. On a concept of a derivative among functions defined on the dyadic field, SIAM J. Math. Anal. **8**, No. 3 (1977), p. 375 - 391.
[13] J. Pál, P. Simon. On a generalization of the concept of derivative, Acta Math. Acad. Sci. Hung., **29**, No. 1-2 (1977), p. 155-164.
[14] F. Schipp. Über einen Ableitungsbegriff von P.L. Butzer und H.J. Wagner, Math. Balkanica. **4** (1974), p. 541-546.

[15] J. Pál, F. Schipp. On the a.e. dyadic differentiability of dyadic integral on R_+, "Theory and applications of Gibbs derivatives". Proc. First Intern. Workshop on Gibbs Derivatives, Sept. 26-28, 1989, Kupari-Dubrovnik. Beograd, Mat. Institute (1989), p. 103-113.
[16] G.H. Hardy, J.L. Littlewood. Some new properties of Fourier constants, Math. Annalen, **97** (1926), p. 159-209.
[17] E. Hille and J.D. Tamarkin. On the absolute integrability of Fourier transforms, Fund. Math., **25** (1935), p. 329-352.
[18] J. Garcia-Cuerva, J.L. Rubio de Francia. Weighted norm inequalities and related topics, North-Holland, Amsterdam - New York - Oxford, 1985.
[19] N.R. Ladhawala. Absolute summability of Walsh-Fourier series, Pacific J. Math., **65** (1976), p. 103 -108.
[20] B.I. Golubov. On an analogue of Hardy's inequality for the Walsh-Fourier transform (in Russian), Izvestiya RAN: Ser. Mat., **65**, No. 3 (2001), p. 3-14.
[21] B.I. Golubov. A modified strong dyadic integral and derivative (in Russian), Mat. sbornik, **193**, No. 4 (2002), p. 37-60.
[22] C.W. Onneweer. Fractional differentiation and Lipschitz spaces on local fields, Trans. Amer. Math. Soc., **258**, No. 1 (1980), p. 155-165.
[23] W.R. Wade. The Gibbs derivative and term by term differentiation of Walsh series, "Theory and applications of Gibbs derivatives". Proc. First Intern. Workshop on Gibbs Derivatives, Sept. 26-28, 1989, Kupari-Dubrovnik. Beograd, Math. Inst. (1989), p. 59-72.
[24] V.A. Skvorčov, W.R. Wade. Generalizations of some results concerning Walsh series and the dyadic derivative, Analysis Math., **5** (1979), p. 249-255.
[25] C.H. Powell, W.R. Wade. Paley sets and term by term differentiation of Walsh series, Acta Math. Acad. Sci. Hung., **62** (1993), No. 1-2, p. 89-96.
[26] F. Schipp. On the dyadic derivative, Acta Math. Acad. Sci. Hung., **28** (1976), p. 145-152.
[27] F. Weisz. Convergence of singular integrals, Ann. Univ. Sci. Budapest, Section Math., **32** (1989), p. 243-256.

Department of Higher Mathematics,
Moscow Engineering Physics Institute,
115409, Moscow, Kashirskoe shosse, 31,
Russia
E-mail address: golubov@mail.mipt.ru