

**CONVERGENCE IN NORM OF THE FEJÉR MEANS ON \mathbb{R}^+
 WITH RESPECT TO WALSH-KACZMARZ SYSTEM**

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ABSTRACT. In this paper we prove that the Fejér integrals of an integrable function f defined on \mathbb{R}^+ converges in norm to the function itself.

Let denote by \mathbf{Z}_2 the discrete cyclic group of order 2, that is $\mathbf{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. Haar measure on \mathbf{Z}_2 is given in the way that the measure of a singleton is $\frac{1}{2}$. Let G be the complete direct product of the countable infinite copies of the compact groups \mathbf{Z}_2 . The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$. The group operation on G is the coordinate-wise addition, the measure and the topology are the product measure and the topology. G is a compact Abelian group, called Walsh group.

For $k \in \mathbf{N}$ and $x \in G$ denote r_k the k -th Rademacher function:

$$r_k(x) := (-1)^{x_k}.$$

Let $n \in \mathbf{N}$, n can be written in the form $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$, that is, n is expressed in the number system based 2. Denote by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. We define the Walsh-Paley system as the set of Walsh-Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{N}).$$

The Walsh-Paley system can be given in the Kaczmarz enumeration as follows:

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-k-1}(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-k-1}},$$

for $x \in G$, $n \in \mathbf{P}$, $\kappa_0(x) := 1(x \in G)$. Define the norm of $x \in G$ by

$$\|x\|_G := \sum_{n \in \mathbf{N}} x_n 2^{-n-1}.$$

The Fine's map $x \mapsto \|x\|_G$ takes G onto $[0, 1)$, the Haar measure to Lebesgue measure on $[0, 1)$, and is 1 – 1 off a countable subset of $[0, 1)$.

The Walsh system in the Kaczmarz enumeration was studied by a lot of authors. A.A. Šneider [10] showed that for the Dirichlet kernel of Walsh-Kaczmarz system the inequality $\limsup_{n \rightarrow \infty} \frac{D_n(x)}{\log n} \geq C > 0$ holds a.e. L.A. Balašov [1] constructed

2000 *Mathematics Subject Classification.* 42C10.

Key words and phrases. Convergence in norm, Fejér means, Fejér kernel, Walsh-Kaczmarz system.

Research supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. M 36511/2001.

an examples for divergent Fourier series. On the other hand F. Schipp [6] and Wo-Sang Young [12] proved that the Walsh-Kaczmarz system is a convergence system, V.A. Skvorcov [9] proved for continuous functions f , that the Fejér means of f converges uniformly to the function f itself. G. Gát [3] proved for integrable functions f , that the Fejér means converges to the function f a.e. and the maximal operator of Fejér means is of type (q, q) for all $1 < q \leq \infty$ and weak type $(1, 1)$. G. Gát and K. Nagy [4] proved that the maximal operator of Fejér means is of type (H, L) .

F. Schipp, W.R. Wade, P. Simon and J. Pál [7] showed the following. Let $f \in L^1(\mathbf{R}^+)$, the Fejér means of f with respect to generalised Walsh-Paley system converges in L^1 -norm to the function f . In this paper we prove the same result for the generalised Walsh-Kaczmarz system.

Let \mathbf{F} denote the set of doubly infinite sequences

$$\{x_n : n \in \mathbf{Z}\}$$

where $x_n = 0$ or 1 ($n \in \mathbf{Z}$) and $\lim_{n \rightarrow -\infty} x_n = 0$. The group operation on \mathbf{F} let be the coordinate-wise addition by modulo 2. $(\mathbf{F}, +)$ is an Abelian group. The norm of $x = (x_n : n \in \mathbf{Z}) \in \mathbf{F}$ is

$$\|x\| := \sum_{n \in \mathbf{Z}} x_n 2^{-n-1}.$$

There is a unique Haar measure μ on \mathbf{F} for which the measure of the unit ball is 1. The Fine's map $x \rightarrow \|x\|$ takes \mathbf{F} onto $\mathbf{R}^+ := [0, \infty)$ and is 1-1 off a countable subset of \mathbf{F} . (The set $\mathbf{Q}^+ := \{p2^q : p \in \mathbf{N}, q \in \mathbf{Z}\}$ represents the dyadic rationals in \mathbf{R}^+ and every point in \mathbf{Q}^+ has two preimages in \mathbf{F} .) At any rate, the Fine's map allows us to identify \mathbf{F} with \mathbf{R}^+ . Define the integer part of an $x \in \mathbf{F}$ by

$$[x] := (\dots, x_{-2}, x_{-1}, 0, 0, \dots)$$

and the fractional part of x by

$$\{x\} := (\dots, 0, 0, x_0, x_1, \dots).$$

Thus $\|[x]\|$ is an integer in \mathbf{R}^+ and $0 \leq \|\{x\}\| \leq 1$ for each $x \in \mathbf{F}$, we shall also denote these corresponding numbers in \mathbf{R}^+ by $[x]$ and $\{x\}$, if no confusion arises. For $x \neq 0$ let

$$|x| := \max\{n \in \mathbf{N} : x_{-n} \neq 0\},$$

that is $2^{|x|-1} \leq [x] < 2^{|x|}$. Denote $\{\psi_y : y \in \mathbf{F}\}$ the set of the characters of the additive group $(\mathbf{F}, +)$. The characters can be generated in the following way. Set

$$\psi_y(x) := (-1)^{\sum_{i+j=-1} x_i y_j}$$

for each $x, y \in \mathbf{F}$. Fix an $n \in \mathbf{N}$ and set

$$\chi_n(x) := \kappa_n(x - [x]) \text{ for all } x \in \mathbf{F}.$$

Define the generalized Walsh-Kaczmarz functions by the equation

$$\chi_y(x) := \chi_{[y]}(x)\chi_{[x]}(y) \quad (x, y \in \mathbf{F}).$$

Let the transformation $\tau_y : \mathbf{F} \rightarrow \mathbf{F}$ be defined by

$$\tau_y(x) := (\dots, x_{-|x|-1}, x_{-1}, x_{-2}, \dots, x_{-|x|}, x_{|y|-1}, x_{|y|-2}, \dots, x_1, x_0, x_{|y|}, \dots)$$

where $(x, y \in \mathbf{F})$ (and two commas show the "centre" of the sequence).

We have

$$\begin{aligned} \chi_y(x) &= \chi_{[y]}(x)\chi_{[x]}(y) = \psi_{[y]}(\tau_y(x))\psi_{[x]}(\tau_x(y)) \\ &= \psi_{[y]}(\tau_y(x))\psi_{[\tau_y(x)]}(y) = \psi_y(\tau_y(x)) \end{aligned}$$

for all $x, y \in \mathbf{F}$.

It is sometimes convenient to work on \mathbf{R}^+ instead of \mathbf{F} , the Fine's map takes Haar measure μ to Lebesgue measure on \mathbf{R}^+ , the characters of \mathbf{F} to generalized Walsh functions on \mathbf{R}^+ . We can leave all notation the same. It is easy to see that

$$\psi_k(x) = \omega_k(x) \text{ and } \chi_k(x) = \kappa_k(x)$$

for $k \in \mathbf{N}$ and $x \in [0, 1)$, where ω_k denotes the k -th Walsh-Paley function and κ_k denotes the k -th Walsh-Kaczmarz function. The functions $\{\psi_j : j \in \mathbf{N}\}$ and $\{\chi_j : j \in \mathbf{N}\}$ form a complete orthonormal system in each interval of the form $[k, k + 1)$, $k = 0, 1, \dots$. For $x \in [k, k + 1)$ we have $\psi_j(x) = \omega_j(x - [x])$ and $\chi_j(x) = \kappa_j(x - [x])$ $j \in \mathbf{N}$. That is ψ_j (χ_j) is a periodic extension of ω_j (κ_j) from $[0, 1)$ to \mathbf{R}^+ .

Define the generalized Dirichlet kernel, the generalized Fejér kernel by

$$D_t^\alpha(y) := \int_0^t \alpha_y(x) dx, \quad \mathcal{K}_u^\alpha(y) := \frac{1}{u} \int_0^u D_t^\alpha(y) dt$$

for $t, y, u \in \mathbf{R}^+$ and $\alpha = \psi$ or χ .

For $t = n \in \mathbf{N}$, the generalized Dirichlet kernel is a zero extension of the Walsh-Paley (Walsh-Kaczmarz)-Dirichlet kernel outside $[0, 1)$, namely

$$D_n^\psi(y) = \begin{cases} \sum_{k=0}^{n-1} \omega_k(y) & 0 \leq y < 1 \\ 0 & y \geq 1, \end{cases} \quad D_n^\chi(y) = \begin{cases} \sum_{k=0}^{n-1} \kappa_k(y) & 0 \leq y < 1 \\ 0 & y \geq 1. \end{cases}$$

(The generalized Fejér kernel is not simply a zero extension of the Walsh-Fejér kernel.) For each $f \in L^1(\mathbf{R}^+)$ we introduce the Walsh-Fourier transform, the Walsh-Dirichlet integrals and the Walsh-Fejér integrals by

$$\hat{f}^\alpha(y) := \int_0^\infty f(x) \alpha_y(x) dx, \quad (S_t^\alpha f)(x) := \int_0^t \hat{f}^\alpha(y) \alpha_y(x) dy, \quad (y, x, t > 0)$$

$$(\sigma_u^\alpha f)(x) := \frac{1}{u} \int_0^u (S_t^\alpha f)(x) dt \quad (u, x > 0),$$

where $\alpha = \psi$ or χ . By definitions and Fubini's theorem we have

$$S_t^\alpha f = f * D_t^\alpha, \quad \sigma_u^\alpha f = f * \mathcal{K}_u^\alpha.$$

Introduce some useful notation, set

$$\mathbf{J}_0(t) := t - [t], \quad \mathbf{J}_l(t) := \int_0^t \chi_l(x) dx \quad (l \in \mathbf{P}, t \in \mathbf{R}^+)$$

and

$$\mathbf{J}_l^{(2)}(t) := \int_0^t \mathbf{J}_l(x) dx \quad (l \in \mathbf{N}, t \in [0, 1)),$$

extend the $\mathbf{J}_l^{(2)}$ to \mathbf{R}^+ by periodicity of period 1.

Lemma 1. For each $u, y > 0$ holds the following equation

$$u \mathcal{K}_u^\chi(y) = \sum_{k=0}^{[u]-1} \left(D_k^\chi(y) + \chi_k(y) \mathbf{J}_{[y]}^{(2)}(1) \right) + (u - [u]) D_{[u]}^\chi(y) + \chi_{[u]}(y) \mathbf{J}_{[y]}^{(2)}(u).$$

Proof. Consequently, by definitions

$$\begin{aligned} D_t^\chi(x) &= \sum_{k=0}^{[t]-1} \int_k^{k+1} \chi_y(x) dy + \int_{[t]}^t \chi_y(x) dy \\ &= \sum_{k=0}^{[t]-1} \chi_k(x) \int_0^1 \chi_{[x]}(y) dy + \chi_{[t]}(x) \int_0^{t-[t]} \chi_{[x]}(y) dy \\ &= D_{[t]}^\chi(x) + \chi_{[t]}(x) \mathbf{J}_{[x]}(t) \end{aligned}$$

for $t, x \in \mathbf{R}^+$ and

$$u\mathcal{K}_u^\chi(y) := \sum_{l=0}^{[u]-1} \int_l^{l+1} D_t^\chi(y) dt + \int_{[u]}^u D_t^\chi(y) dt$$

for $u, y \in \mathbf{R}^+$. These imply that

$$\int_{[u]}^u D_t^\chi(y) dt = (u - [u])D_{[u]}^\chi(y) + \chi_{[u]}(y)\mathbf{J}_{[y]}^{(2)}(u)$$

and

$$\begin{aligned} \int_l^{l+1} D_t^\chi(y) dt &= \int_l^{l+1} D_{[t]}^\chi(y) + \chi_{[t]}(y)\mathbf{J}_{[y]}(t) dt \\ &= \int_l^{l+1} \sum_{k=0}^{[t]-1} \chi_k(y) \int_0^1 \chi_{[y]}(s) ds dt + \chi_l(y) \int_l^{l+1} \mathbf{J}_{[y]}(t) dt \\ &= D_l^\chi(y) + \chi_l(y)\mathbf{J}_{[y]}^{(2)}(1). \end{aligned}$$

This completes the proof of Lemma 1. \square

Lemma 2 (V.A. Skvorcov). *There exists a constant $C > 0$ such that*

$$\|K_n^\kappa\|_1 \leq C \quad (n \in \mathbf{N}).$$

Lemma 3. *There exists a constant $C > 0$ such that*

$$\|\mathcal{K}_u^\chi\|_1 \leq C, \quad (u \geq 1).$$

Proof. It follows from Lemma 1 that

$$|\mathcal{K}_u^\chi(y)| \leq |K_{[u]}^\kappa(y)| + |\mathbf{J}_{[y]}^{(2)}(1)| + \left| \frac{1}{u} D_{[u]}^\chi(y) \right| + \left| \frac{1}{u} \mathbf{J}_{[y]}^{(2)}(u) \right| \quad (u \geq 1, y \in \mathbf{R}^+).$$

By Skvorcov's lemma, there exists a $C > 0$ such that

$$\|K_{[u]}^\kappa\| \leq C \quad (u \geq 1).$$

By the fact, that for integer values of t , the generalized Dirichlet kernel D_t^χ is a zero extension of the Walsh-Kaczmarz-Dirichlet kernel outside $[0, 1]$, we have that

$$\left\| \frac{1}{u} D_{[u]}^\chi \right\| \leq 1.$$

Let $2^n \leq l < 2^{n+1}$, $t \in \mathbf{R}^+$ and denote $\bar{t} := \{t\}$, by definitions

$$\begin{aligned} \mathbf{J}_l(t) &= \int_{[t]}^t \chi_l(x) dx = \int_0^{\{t\}} \chi_l(x) dx = \int_{\frac{\bar{t}_0}{2} + \dots + \frac{\bar{t}_{n-1}}{2^n}}^{\bar{t}} \chi_l(x) dx \\ &= (-1)^{\sum_{k=0}^{n-1} l_k \bar{t}_{n-1-k}} \int_{\frac{\bar{t}_0}{2} + \dots + \frac{\bar{t}_{n-1}}{2^n}}^{\bar{t}} (-1)^{x_n} dx \\ &= \chi_l\left(\frac{\bar{t}_0}{2} + \dots + \frac{\bar{t}_{n-1}}{2^n}\right) \left[\frac{\bar{t}_n}{2^{n+1}} + (-1)^{\bar{t}_n} \left(\frac{\bar{t}_{n+1}}{2^{n+2}} + \frac{\bar{t}_{n+2}}{2^{n+3}} + \dots \right) \right]. \end{aligned}$$

Consequently,

$$|\mathbf{J}_l(t)| \leq 2^{-n}$$

for $2^n \leq l < 2^{n+1}$, $n \in \mathbf{N}$ and $t \in \mathbf{R}^+$.

Investigate the second integral $\mathbf{J}_l^{(2)}(1)$. Let $l = 2^n$

$$|\mathbf{J}_l^{(2)}(1)| \leq \int_0^1 |\mathbf{J}_l(t)| dt \leq 2^{-n},$$

and let $l = 2^n + 2^m + k$ ($0 \leq m < n$, $0 \leq k < 2^m$)

$$\mathbf{J}_l^{(2)}(1) =$$

$$\begin{aligned}
 &= \int_0^1 (-1)^{\sum_{j=0}^{m-1} l_j t_{n-1-j} + t_{n-1-m}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n} \left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots \right) \right] dt \\
 &= \int_{\{x \in [0,1]: x_{n-m-1}=0\}} (-1)^{\sum_{j=0}^{m-1} l_j t_{n-1-j}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n} \left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots \right) \right] dt \\
 &\quad - \int_{\{x \in [0,1]: x_{n-m-1}=1\}} (-1)^{\sum_{j=0}^{m-1} l_j t_{n-1-j}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n} \left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots \right) \right] dt \\
 &= 0.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \|\mathbf{J}_{[1]}^{(2)}(1)\|_1 &= \int_0^\infty |\mathbf{J}_{[y]}^{(2)}(1)| dy = \int_0^1 |\mathbf{J}_0^{(2)}(1)| dy + \sum_{i=1}^\infty \int_i^{i+1} |\mathbf{J}_i^{(2)}(1)| dy \\
 &\leq \frac{1}{2} + \sum_{n=1}^\infty 2^{-n} = \frac{3}{2}.
 \end{aligned}$$

Last, we have to see the L^1 -norm of $\frac{1}{u} \mathbf{J}_{[1]}^{(2)}(u)$ for $u \geq 1$.

$$\begin{aligned}
 \frac{1}{u} \int_0^\infty |\mathbf{J}_{[y]}^{(2)}(u)| dy &= \frac{1}{u} \int_0^1 |\mathbf{J}_0^{(2)}(u)| dy + \frac{1}{u} \int_1^4 |\mathbf{J}_0^{(2)}(u)| dy \\
 &\quad + \frac{1}{u} \int_4^\infty |\mathbf{J}_{[y]}^{(2)}(u)| dy =: I_1(u) + I_2(u) + I_3(u).
 \end{aligned}$$

Using the periodicity of $\mathbf{J}_0^{(2)}$, it is easy to estimate the first integral

$$\mathbf{J}_0^{(2)}(u) = \int_0^{u-[u]} t - [t] dt = \frac{(u - [u])^2}{2}, \text{ and } I_1(u) = \left| \frac{(u - [u])^2}{2u} \right| \leq \frac{1}{2}$$

for $u \geq 1$.

Similarly, the second integral $I_2(u) \leq c$.

Now we will estimate the third integral $I_3(u)$. Let $l := 2^n$, by the periodicity of $\mathbf{J}_l^{(2)}$ and by the inequality on \mathbf{J}_l we conclude that

$$|\mathbf{J}_l^{(2)}(u)| = \int_0^{u-[u]} 2^{-n} dt \leq 2^{-n}.$$

Let $l := 2^n + 2^m + k$, where $m := \min\{i : l_i \neq 0\}$ and $2^m < k < 2^n - 1$. Set

$$z^0 := (\dots, 0, 0, z_0, z_1, \dots, z_{n-m-2}, 0, z_{n-m}, z_{n-m+1}, \dots)$$

and

$$z^1 := (\dots, 0, 0, z_0, z_1, \dots, z_{n-m-2}, 1, z_{n-m}, z_{n-m+1}, \dots)$$

for any $z \in [0, 1)$.

$$\begin{aligned}
 &\int_{I_{n-m}(z^0)} (-1)^{\sum_{j=m+1}^{n-1} l_j t_{n-1-j} + t_{n-1-m}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n} \left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots \right) \right] dt \\
 &+ \int_{I_{n-m}(z^1)} (-1)^{\sum_{j=m+1}^{n-1} l_j t_{n-1-j} + t_{n-1-m}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n} \left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots \right) \right] dt \\
 &= 0.
 \end{aligned}$$

That is

$$\int_{I_{n-m}(z^0) \cup I_{n-m}(z^1)} \mathbf{J}_l(t) dt = 0$$

for any $z \in [0, 1)$. Therefore

$$\mathbf{J}_l^{(2)}(u) = \int_0^{\{u\}} \mathbf{J}_l(t) dt = \int_{\frac{\bar{u}_0}{2} + \dots + \frac{\bar{u}_{n-m-1}}{2^{n-m}}}^{\bar{u}} \mathbf{J}_l(t) dt$$

and

$$|J_l^{(2)}(u)| \leq 2^{-n} 2^{-(n-m)},$$

where $\bar{u} := \{u\}$.

$$\begin{aligned} & \int_4^\infty |J_{[y]}^{(2)}(u)| dy = \sum_{n=2}^\infty \int_{2^n}^{2^{n+1}} |J_{[y]}^{(2)}(u)| dy \\ &= \sum_{n=2}^\infty \left[\int_{2^n}^{2^{n+1}} |J_{[y]}^{(2)}(u)| dy + \sum_{\{k:k_0=0\}} \int_{2^{n+1+k}}^{2^{n+k+2}} |J_{[y]}^{(2)}(u)| dy \right. \\ &+ \sum_{\{k:k_0=k_1=0\}} \int_{2^{n+2^1+k}}^{2^{n+2^1+k+1}} |J_{[y]}^{(2)}(u)| dy + \dots \\ &+ \left. \sum_{\{k:k_i=0, l \leq n-1\}} \int_{2^{n+2^{n-1}+k}}^{2^{n+2^{n-1}+k+1}} |J_{[y]}^{(2)}(u)| dy \right] \\ &\leq \sum_{n=2}^\infty \left[\frac{1}{2^n} + \frac{1}{2^n} \frac{1}{2^n} 2^{n-1} + \frac{1}{2^n} \frac{1}{2^{n-1}} 2^{n-2} + \dots + \frac{1}{2^n} \frac{1}{2^1} 2^0 \right] \\ &= c + \frac{1}{2} \sum_{n=2}^\infty \frac{n}{2^n} \leq c. \end{aligned}$$

The proof is complete. □

By standard argument the following theorem holds.

Theorem 1. For every $f \in L^1(\mathbf{R}^+)$ holds

$$\lim_{u \rightarrow \infty} \|\sigma_u^\chi f - f\|_1 = 0.$$

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