

REMARKS ON GENERALIZED RAMANUJAN SUMS AND EVEN FUNCTIONS

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Dedicated to Professor W.R. Wade on his 60th birthday

ABSTRACT. We prove a simple formula for the main value of r -even functions and give applications of it. Considering the generalized Ramanujan sums $c_A(n, r)$ involving regular systems A of divisors we show that it is not possible to develop a Fourier theory with respect to $c_A(n, r)$, like in the usual case of classical Ramanujan sums $c(n, r)$.

1. INTRODUCTION

Ramanujan's trigonometric sum $c(n, r)$ is defined as the sum of n -th powers of the r -th primitive roots of unity, i.e.

$$c(n, r) = \sum_{\substack{k \pmod{r} \\ (k, r) = 1}} \exp(2\pi i k n / r),$$

where $r, n \in \mathbb{N} \equiv \{1, 2, 3, \dots\}$. In his original paper [R18] S. Ramanujan proved, among others, that for every $n \in \mathbb{N}$,

$$\begin{aligned} (1) \quad \frac{\sigma(n)}{n} &= \frac{\pi^2}{6} \sum_{r=1}^{\infty} \frac{c(n, r)}{r^2} \\ &= \frac{\pi^2}{6} \left(1 + \frac{(-1)^n}{2^2} + \frac{2 \cos(2\pi n/3)}{3^2} + \frac{2 \cos(\pi n/2)}{4^2} + \dots \right), \end{aligned}$$

where $\sigma(n)$ stands for the sum of the positive divisors of n . Formula (1) shows how the values of $\sigma(n)/n$ fluctuate harmonically about their mean value $\pi^2/6$. Here the main value of a function $f: \mathbb{N} \rightarrow \mathbb{C}$ is defined by $M(f) = \lim_{x \rightarrow \infty} \sum_{n \leq x} f(n)$, if this limit exists.

The orthogonality relations

$$(2) \quad M(c(\cdot, r)c(\cdot, s)) = \delta_{r,s} \phi(r),$$

where $\delta_{r,s}$ is the Kronecker-symbol and $\phi(r) = c(r, r)$ is Euler's arithmetical function, suggest to have expansions, convergent pointwise or in other sense, of functions f of the form

$$f(n) = \sum_{r=1}^{\infty} a_r c(n, r), \quad n \in \mathbb{N},$$

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where the coefficients a_r are given by

$$a_r = \frac{1}{\phi(r)} M(fc_r).$$

A Fourier analysis of arithmetical functions, with respect to Ramanujan sums, parallel to periodic and almost periodic functions, was developed by several authors, cf. J. Delsarte [D45], W. Schwarz and J. Spilker [SchS74], A. Hildebrand [H84], G. Gát [G91], L. Lucht [L95], see also the books [K75], [SchS94].

Ramanujan's sum $c(n, r)$ is an example of an r -even function (even function (mod r)), i.e. a function $f(n, r)$ such that $f(n, r) = f(gcd(n, r), r)$ for every $n, r \in \mathbb{N}$. This concept was introduced by E. Cohen [C55]. For a fixed r the set $\mathcal{E}_r$ of r -even functions $f: \mathbb{N} \rightarrow \mathbb{C}$ is a complex Hilbert space of finite dimension $\tau(r)$ = the number of divisors of r , under the inner product

$$\langle f, g \rangle = \frac{1}{r} \sum_{d|r} \phi(d) f(r/d) \overline{g(r/d)},$$

and $(c'(\cdot, q))_{q|r}$, $c'(n, q) = \frac{1}{\sqrt{\phi(q)}} c(n, q)$ is an orthonormal basis for $\mathcal{E}_r$. The main value $M(c(\cdot, r))$ exists for every $r \geq 1$, it is $M(c(\cdot, r)) = \delta_{r,1}$, hence $M(f)$ exists for each $f \in \mathcal{E}_r$. If $\mathcal{E} = \cup_{r \in \mathbb{N}} \mathcal{E}_r$, then $\mathcal{E}$ is a dense subalgebra of the algebra $\mathbb{C}^{\mathbb{N}}$.

For these and various other properties of $c(n, r)$ and of r -even functions see [K75], [McC86], [SchS94].

The Ramanujan sum $c(n, r)$ has been generalized in several directions. One of the generalizations, due to P.J. McCarthy [McC68], notation $c_A(n, r)$, is involving regular systems A of divisors, see Section 2, and it has all nice algebraic properties of the usual kind.

The following question can be formulated. Is it possible to develop a Fourier theory concerning the generalized sums $c_A(n, r)$, analogous to the usual one?

The aims of this paper are the following:

- to prove a simple formula for the main value of r -even functions (Proposition 1), this result seems to have not been appeared in the literature, and to give applications of it,
- to compute the main value of $c_A(\cdot, r)$ for an arbitrary regular system A (Proposition 2),
- to show that the answer is negative for the question formulated above (Propositions 3 and 4).

2. REGULAR CONVOLUTIONS

Let $A(n)$ be a subset of the set of positive divisors of n for each $n \in \mathbb{N}$. The A -convolution of the functions $f, g: \mathbb{N} \rightarrow \mathbb{C}$ is given by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g(n/d), \quad n \in \mathbb{N}.$$

The system $A = (A(n))_{n \in \mathbb{N}}$ of divisors is called regular, cf. [N63], if

- (a) $(\mathbb{C}^{\mathbb{N}}, +, *_A)$ is a commutative ring with unity,
- (b) the A -convolution of multiplicative functions is multiplicative (recall that function f is multiplicative if $f(mn) = f(m)f(n)$ whenever $gcd(m, n) = 1$),
- (c) the constant 1 function has an inverse μ_A (generalized Möbius function) with respect to $*_A$ and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power p^a ($a \geq 1$).

It can be shown, cf. [N63], [McC86], that $*_A$ is regular iff

- (i) $A(mn) = \{de : d \in A(m), e \in A(n)\}$ for every $m, n \in \mathbb{N}$, $(m, n) = 1$,

- (ii) for every prime power p^a ($a \geq 1$) there exists a divisor $t = t_A(p^a)$ of a , called the type of p^a with respect to A , such that $A(p^{it}) = \{1, p^t, p^{2t}, \dots, p^{it}\}$ for every $i \in \{0, 1, \dots, a/t\}$.

Examples of regular systems of divisors are $A = D$, where $D(n)$ is the set of all positive divisors of n , and $A = U$, where $U(n)$ is the set of divisors d of n such that $(d, n/d) = 1$ (unitary divisors). For every prime power p^a one has $t_D(p^a) = 1$ and $t_U(p^a) = a$. Here $*_D$ and $*_U$ are the Dirichlet convolution and the unitary convolution, respectively. For properties of regular convolutions and related arithmetical functions we refer to [N63], [McC86], [S78], [T97].

The following generalization of $c(n, r)$ is due to P.J. McCarthy [McC68], see also [McC86]. For a regular system A of divisors and $r, n \in \mathbb{N}$ let

$$c_A(n, r) = \sum_{\substack{k \pmod{r} \\ (k, r)_A = 1}} \exp(2\pi i k n / r),$$

where $(k, r)_A = \max\{d \in \mathbb{N} : d|k, d \in A(r)\}$, and let $c_A(r, r) \equiv \phi_A(r)$ be the generalized Euler function. For $A = U$ the functions $c_U(n, r) \equiv c^*(n, r)$ and $\phi_U(r) \equiv \phi^*(r)$ were introduced by E. Cohen [C60].

$c_A(n, r)$ preserves the basic properties of $c(n, r)$. For example, for every regular A and $r, n \in \mathbb{N}$ one has

$$(3) \quad c_A(n, r) = \sum_{d|n, d \in A(r)} d \mu_A(r/d),$$

hence $c_A(n, r)$ is integer-valued and it is multiplicative in r . Note that

$$(4) \quad c_A(n, r) = \sum_{d|r, \gamma_A(r)|d} c(n, d),$$

where γ_A is multiplicative and $\gamma_A(p^a) = p^{a-t+1}$ for every prime power p^a ($a \geq 1$), here $t = t_A(p^a)$ (generalized core function), see [McC68], Th. 2.

The function f is called A -even (mod r) if $f(n, r) = f((n, r)_A, r)$ for each $n, r \in \mathbb{N}$, cf. [McC68]. Let $\mathcal{E}_{A,r}$ denote the set of functions $f(n, r)$ which are A -even (mod r). Then $\mathcal{E}_{A,r} \subset \mathcal{E}_r$ for every A and $r \in \mathbb{N}$. For example, $c_A(n, r)$ is A -even (mod r). Let $\mathcal{E}_A = \cup_{r \in \mathbb{N}} \mathcal{E}_{A,r}$.

3. RESULTS

Proposition 1. *Let $r \in \mathbb{N}$ and $f \in \mathcal{E}_r$. Then*

$$M(f) = \frac{1}{r} (f *_D \phi)(r) \equiv \frac{1}{r} \sum_{e|r} f(e) \phi(r/e).$$

Moreover, for every $x \geq 1$ and every $\varepsilon > 0$,

$$\frac{1}{K_f} \left| \sum_{n \leq x} f(n) - M(f)x \right| \leq C_\varepsilon r^{1+\varepsilon},$$

where $|f(n)| \leq K_f$, $n \in \mathbb{N}$ and C_ε is a constant depending only on ε .

Proof. Write f in the form

$$f(n) = \sum_{q|r} h(q) c(n, q), \quad n \in \mathbb{N},$$

where the Fourier coefficients $h(q)$, are given for every $q|r$ by

$$h(q) = \frac{1}{r \phi(q)} \sum_{e|r} \phi(e) f(r/e) c(r/e, q) = \frac{1}{r} \sum_{e|r} f(r/e) c(r/q, e),$$

and use that $\sum_{n \leq x} c_q(n) = \delta_{q,1}x + R_q(x)$, where $|R_q(x)| \leq q^{1+\varepsilon}$, cf. [McC86], Ch. 2, [K75], Ch. 7. Then

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{n \leq x} \sum_{q|r} h(q) c(n, q) = \sum_{q|r} h(q) \sum_{n \leq x} c(n, q) \\ &= \sum_{q|r} h(q) (\delta_{q,1}x + R_q(x)) = h(1)x + \sum_{q|r} h(q) R_q(x), \end{aligned}$$

where $h(1) = \frac{1}{r}(f * \phi)(r)$, $|h(q)| \leq \frac{1}{r} \sum_{e|r} |f(r/e)| |c(r/q, e)| \leq K_f \frac{1}{r} \sum_{e|r} e = K_f \sigma(r)/r$ and

$$\left| \sum_{q|r} h(q) R_q(x) \right| \leq K_f (\sigma(r)/r) \sum_{q|r} q^{1+\varepsilon}$$

and the result follows by the usual estimates $\sigma(r) \leq r\tau(r)$ and $\tau(r) \ll r^\varepsilon$. $\square$

As an application, consider the function $\phi(s, d, n)$ defined by $\phi(s, d, n) = \#\{k \in \mathbb{N} \cap [1, n] : (s + (k-1)d, n) = 1\}$, where $s, d \in \mathbb{N}$, $(s, d) = 1$. Note that $\phi(1, 1, n) = \phi(n)$ is the Euler function. T. Maxsein [M90] pointed out that $\phi(s, \cdot, n)$ is an n -even function and determined its main value:

$$M(\phi(s, \cdot, n)) = n \prod_{p|n} (1 - 1/p + 1/p^2),$$

where the product is over the prime divisors p of n . This follows from Proposition 1 by easy computations.

Proposition 1 applies also for $f(n) = c_A(n, r)$, which is an r -even function. However, a better error term can be obtained and the computations are simpler by a direct proof using representation (3). We have

Proposition 2. *For every regular system A and $r \in \mathbb{N}$,*

$$M(c_A(\cdot, r)) = \delta_{r,1}$$

and

$$\left| \sum_{n \leq x} c_A(n, r) - \delta_{r,1}x \right| \leq \psi_A(r),$$

where ψ_A is multiplicative and $\psi_A(p^a) = p^a + p^{a-t}$ for every prime power p^a ($a \geq 1$), where $t = t_A(p^a)$ (generalized Dedekind function).

Proof. Using (3),

$$\begin{aligned} \sum_{n \leq x} c_A(n, r) &= \sum_{\substack{n \leq x \\ d|n, d \in A(r)}} d\mu_A(r/d) = \sum_{d \in A(r)} d\mu_A(r/d) \sum_{\substack{n \leq x \\ d|n}} 1 = \sum_{d \in A(r)} d\mu_A(r/d) [x/d] \\ &= x \sum_{d \in A(r)} \mu_A(r/d) - \sum_{d \in A(r)} d\mu_A(r/d) (x/d - [x/d]) \\ &= x\delta_{r,1} + R_A(r), \end{aligned}$$

where

$$|R_A(r)| \leq \sum_{d \in A(r)} d |\mu_A(r/d)| = \psi_A(r).$$

$\square$

Note that $\psi_A(r) \leq \sigma(r) < Cr \ln \ln r$ for every $r \in \mathbb{N}$, with a suitable constant C .

The following result shows that for every system $A \neq D$ the orthogonality relations (2) are violated.

Proposition 3. *For every regular system A ,*

$$M(c_A(\cdot, r)c_A(\cdot, s)) = \begin{cases} \phi_A(r), & \text{if } r = s, \\ 0, & \text{if } rs > 1, (r, s) = 1, \end{cases}$$

but for $A \neq D$ there exist r, s such that $r \neq s$ and $M(c_A(\cdot, r)c_A(\cdot, s)) \neq 0$.

Proof. Let A be arbitrary. Applying (4) and (2) we obtain

$$(5) \quad M(c_A(\cdot, r)c_A(\cdot, s)) = \sum_{\substack{d|r, \gamma_A(r)|d \\ e|s, \gamma_A(s)|e}} M(c(\cdot, r)c(\cdot, s)) = \sum_{\substack{d|r, \gamma_A(r)|d \\ d|s, \gamma_A(s)|d}} \phi(d).$$

Using that $\gamma_A(k) > 1$ for $k > 1$ we get the first part of the desired result.

Now let $A \neq D$. Then there exists a prime power p^a such that $t \equiv t_A(p^a) > 1$. Therefore $A(p^t) = \{1, p^t\}$. Let $r = p$, $s = p^t$, then $r \neq s$ and the last sum in (5) has one single term, namely $\phi(p) = p - 1 \neq 0$. $\square$

Proposition 4. $\mathcal{E}_A$ *is a vector space if and only if $A = D$.*

Proof. We show that $\mathcal{E}_A$ is not a vector space for $A \neq D$.

Suppose that $A \neq D$. Then there exists a prime power p^a such that $t \equiv t_A(p^a) > 1$. Hence $A(p^t) = \{1, p^t\}$. Let

$$f(n) = (n, p)_A = \begin{cases} p, & \text{if } p|n, \\ 1, & \text{otherwise,} \end{cases}$$

$$g(n) = (n, p^t)_A = \begin{cases} p^t, & \text{if } p^t|n, \\ 1, & \text{otherwise.} \end{cases}$$

Then $f, g \in \mathcal{E}_A$ and suppose that $h = f + g \in \mathcal{E}_A$, i.e. there exists $r \in \mathbb{N}$ such that $f + g \in \mathcal{E}_{A, r}$. Here

$$h(n) = f(n) + g(n) = \begin{cases} p + p^t, & \text{if } p^t|n, \\ 1 + p, & \text{if } p|n, p^t \nmid n, \\ 2, & \text{if } p \nmid n. \end{cases}$$

From $1 + p = h(p) = h((p, r)_A)$ we have $(p, r)_A = p$, $p \in A(r)$ and from $p + p^t = h(p^t) = h((p^t, r)_A)$ we obtain $(p^t, r)_A = p^t$, $p^t \in A(r)$.

Let $r = p^k s$, where $p \nmid s$. Then $p \in A(p^k)$ and $p^t \in A(p^k)$, therefore $p \in A(p^t)$, in contradiction with $A(p^t) = \{1, p^t\}$. $\square$

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