

The contracted model of exploded real numbers

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ABSTRACT. In this paper we show that a set of complex numbers u , where $\operatorname{Im} u = \frac{1}{2} \cdot \frac{n}{|n|+1}$, ($n = 0, \pm 1, \pm 2, \dots$) and

$$(\operatorname{Re} u) \cdot (\operatorname{Im} u) \geq 0$$

is one of the suitable model of exploded real numbers. This model allows the conclusion that the set of exploded real numbers exists.

In [1] we introduced the set of exploded real numbers $\overset{\sqcup}{R}$ with the following postulates and requirements.

Postulate of extension:

The set of real numbers is a proper subset of exploded real numbers. For any real number x there exists one exploded real number which is called exploded x or the exploded of x . Moreover, the set of exploded x is called the set of exploded real numbers.

Postulate of unambiguity:

For any pair of real numbers x and y , their explodeds are equal if and only if x is equal to y .

Postulate of ordering:

For any pair of real numbers x and y , the exploded x is less than exploded y if and only if x is less than y .

Postulate of super-addition:

For any pair of real numbers x and y , the super-sum of their explodeds is exploded of their sum.

Postulate of super-multiplication:

For any pair of real numbers x and y , the super-product of their explodeds is the exploded of their product.

Requirement of equality for exploded real numbers:

If x and y are real numbers then x as an exploded real number equals to y as an exploded real number if they are equal in the traditional sense.

Requirement of ordering for exploded real numbers:

If x and y are real numbers then x as an exploded real number is less than y as an exploded real number if x is less than y in the traditional sense.

Requirement of monotonicity of super-addition:

If u and v are arbitrary exploded real numbers and u is less than v then, for any exploded real number w , u superplus w is less than v superplus w .

Requirement of monotonicity of super-multiplication:

If u and v are arbitrary exploded real numbers and u is less than v then, for any positive exploded real number w , u super- multiplied by w is less than v super-multiplied by w .

Definition 1. The explosion of real numbers in a contracted sense: for any real number x , its exploded is

$$(1.1) \quad \overset{\sqcup}{x} = (\operatorname{sgn} x) \left(\operatorname{area} \operatorname{th} \{|x|\} + \frac{i}{2} \frac{[x]}{[x] + 1} \right), \quad x \in R.$$

2000 Mathematics Subject Classification. 03C30.

Key words and phrases. Exploded numbers, explosion of real numbers, compression of exploded numbers, compressed numbers, super- multiplication, super-addition.

Clearly,

$$\operatorname{Im} \bar{x} = \frac{1}{2} \frac{n}{|n|+1}, \quad \text{where } n \text{ is an integer number and } (\operatorname{Re} \bar{x}) \cdot (\operatorname{Im} \bar{x}) \geq 0.$$

Theorem 2. The mapping $x \rightarrow \bar{x}$ is mutually unambiguous.

Proof. Obviously, if $x = y \Rightarrow \bar{x} = \bar{y}$ ($\operatorname{Re} \bar{x} = \operatorname{Re} \bar{y}$ and $\operatorname{Im} \bar{x} = \operatorname{Im} \bar{y}$)
Conversely, we assume that $\bar{x} = \bar{y}$. Hence,

$$(2.1) \quad (\operatorname{sgn} x) \operatorname{area th}\{|x|\} = (\operatorname{sgn} y) \operatorname{area th}\{|y|\}$$

and

$$(2.2) \quad (\operatorname{sgn} x) \frac{[|x|]}{[|x|]+1} = (\operatorname{sgn} y) \frac{[|y|]}{[|y|]+1}.$$

By (2.2) the cases $|x| \geq 1$ and $|y| < 1$; $|x| < 1$ and $|y| \geq 1$ are not allowed so we have the following two cases

$$a) \quad 0 \leq |x|, |y| < 1$$

or

$$b) \quad |x|, |y| \geq 1,$$

only.

In the case a) exception of $x = y = 0$, $|x| < 1$ and $y = 0$; $x = 0$ and $|y| < 1$ is not allowed. (See (2.1).)
Otherwise we can see that $\{|x|\}$ and $\{|y|\}$ are positive numbers, so (2.1) gives that $\operatorname{sgn} x = \operatorname{sgn} y$.

In the case b) we have that $\frac{[|x|]}{[|x|]+1}$ and $\frac{[|y|]}{[|y|]+1}$ are positive numbers so, (2.2) gives that $\operatorname{sgn} x = \operatorname{sgn} y$.

Collecting these, for all allowed cases of the pairs x, y we obtain

$$(2.3) \quad \operatorname{sgn} x = \operatorname{sgn} y.$$

Using (2.3) we can see that (2.2) yields

$$(2.4) \quad [|x|] = [|y|].$$

Using (2.3) again by (2.1) we get

$$(2.5) \quad \{|x|\} = \{|y|\}.$$

By (2.4) and (2.5) we have that $|x| = |y|$ and finally (2.3) gives that $x = y$. ■

Remark. Theorem 2 shows that the Postulate of unambiguity is fulfilled.

Theorem 3. If u is a complex number such that $\operatorname{Im} u = \frac{1}{2} \frac{n}{|n|+1}$, $n = 0, \pm 1, \pm 2, \dots$ and $(\operatorname{Re} u) \cdot (\operatorname{Im} u) \geq 0$, then

$$\overline{\frac{\operatorname{Im} u}{\frac{1}{2} - |\operatorname{Im} u|} + \operatorname{th} \operatorname{Re} u} = u.$$

Proof. It is easy to see that

$$(3.2) \quad \frac{\operatorname{Im} u}{\frac{1}{2} - |\operatorname{Im} u|} = n$$

is valid. First let be $n = 1, 2, 3, \dots$ Now we have that $\operatorname{Re} u \geq 0$ and by (1.1)

$$\overline{n + \operatorname{th} \operatorname{Re} u} = \operatorname{area th}(\operatorname{th} \operatorname{Re} u) + \frac{i}{2} \frac{n}{n+1} = \operatorname{Re} u + i \operatorname{Im} u = u.$$

For $n = 0$, u is a real number, so $\operatorname{Re} u = u$. Using (1.1) we have:

$$\begin{aligned}\overline{\operatorname{th} u} &= (\operatorname{sgn} u) \operatorname{area} \operatorname{th}\{|\operatorname{th} u|\} = (\operatorname{sgn} u) \operatorname{area} \operatorname{th} |\operatorname{th} u| = \\ &= (\operatorname{sgn} u) \operatorname{area} \operatorname{th}(\operatorname{th} |u|) = (\operatorname{sgn} u)|u| = u.\end{aligned}$$

Finally, for $n = -1, -2, -3$ we have that $\operatorname{Re} u \leq 0$ and by (1.1)

$$\overline{n + \operatorname{th} \operatorname{Re} u} = -\left(\operatorname{area} \operatorname{th}(-\operatorname{th} \operatorname{Re} u) + \frac{i}{2} \frac{|n|}{|n|+1}\right) = \operatorname{Re} u + i \operatorname{Im} u = u.$$

■

Theorem 3 and (1.2) yield

Corollary 4. The complex number u is an exploded real number in a contracted sense, if and only if $\operatorname{Im} u = \frac{1}{2} \frac{n}{|n|+1}$, $n = 0, \pm 1, \pm 2, \dots$, and $(\operatorname{Re} u) \cdot (\operatorname{Im} u) \geq 0$.

We denote the set of exploded real numbers, in a contracted sense, by $\overline{R}$.

$$\overline{R} = \{u \in \mathbf{C} : u = \operatorname{Re} u + i \operatorname{Im} u, \operatorname{Im} u = \frac{1}{2} \frac{n}{|n|+1}, n \text{ is integer and } (\operatorname{Re} u) \cdot (\operatorname{Im} u) \geq 0.\}$$

Definition 5. For any set $S \subseteq \mathbf{R}$, the exploded S is: $\overline{S} = \{u \in \mathbf{C} : u = \overline{x} \text{ such that } x \in S\}$. Considering the open interval $(-1, 1)$ by Definitions 1 and 5 we obtain

Corollary 6. $\overline{(-1, 1)} = \mathbf{R}$.

So, we can see that the Postulate of extension is fulfilled.

Definition 7. The compression of exploded real numbers: for any exploded real number u , its compressed is

$$(7.1) \quad \underline{u} = \frac{\operatorname{Im} u}{\frac{1}{2} - |\operatorname{Im} u|} + \operatorname{th} \operatorname{Re} u, \quad u \in \overline{R}.$$

By (3.1) and (7.1) we have the identity

$$(7.2) \quad \overline{\left(\underline{u}\right)} = u, \quad u \in \overline{R}.$$

Definition 8. For set $S \subseteq \mathbf{R}$, the compressed of S is: $\underline{S} = \{x \in \mathbf{R} : x = \underline{u}, \text{ such that } u \in S\}$.

Theorem 9. For any real number x the identity

$$(9.1) \quad \overline{\left(\underline{x}\right)} = x, \quad x \in \mathbf{R}$$

holds.

Definitions 5 and 8 with (7.2) and (9.1) yield

Corollary 10.

$$(10.1) \quad \overline{\left(\underline{S}\right)} = S, \quad S \subseteq \mathbf{R}$$

and

$$(10.2) \quad \overline{\left(\underline{S}\right)} = S, \quad S \subseteq \overline{R}.$$

Definition 11. For any $x, y \in R$ we say that $\overset{\square}{x} < \overset{\square}{y}$ if $\text{Im } \overset{\square}{x} < \text{Im } \overset{\square}{y}$ or if $\text{Im } \overset{\square}{x} = \text{Im } \overset{\square}{y}$ then $\text{Re } \overset{\square}{x} < \text{Re } \overset{\square}{y}$.

Definition 12. For any $x, y \in R$ we say that $\overset{\square}{x} > \overset{\square}{y}$ if $\overset{\square}{y} < \overset{\square}{x}$.

Theorem 13. For any x the inequality $\overset{\square}{x} < \overset{\square}{y}$ holds if and only if $x < y$.

Proof.

Necessity. Let us assume that $\overset{\square}{x} < \overset{\square}{y}$. By Definition 11 we consider two cases:

Case 1. $\text{Im } \overset{\square}{x} < \text{Im } \overset{\square}{y}$, that is, by (1.1) we have

$$(13.1) \quad (\text{sgn } x) \frac{[|x|]}{[|x|] + 1} < (\text{sgn } y) \frac{[|y|]}{[|y|] + 1}$$

Now, if $x \geq y$ then considering the monotonicity of the function $f(x) = (\text{sgn } x) \frac{[|x|]}{[|x|] + 1}$ we have that $f(x) \geq f(y)$ which contradicts (13.1). So, $x < y$.

Case 2. $\text{Im } \overset{\square}{x} = \text{Im } \overset{\square}{y}$ and $\text{Re } \overset{\square}{x} < \text{Re } \overset{\square}{y}$. Now we have (2.2) and

$$(13.2) \quad (\text{sgn } x) \text{area th}\{|x|\} < (\text{sgn } y) \text{area th}\{|y|\}$$

moreover, x and y are not integer numbers. If $x = 0$ then $y > 0$, if $y = 0$ then $x < 0$. Otherwise, $\text{area th}\{|x|\}, \text{area th}\{|y|\} > 0$. Inequality $\text{sgn } x > \text{sgn } y$ is not allowed.

If $\text{sgn } x < \text{sgn } y$ then $x < y$, obviously.

If $\text{sgn } x = \text{sgn } y = 1$, then (2.2) yields that $[x] = [y]$ and (13.2) gives that $\{x\} < \{y\}$, so $0 < x < y$.

If $\text{sgn } x = \text{sgn } y = -1$, then (2.2) yields that $[|x|] = [|y|]$ and the identity $[|x|] = -([x] + 1)$ shows that $[x] = [y]$. Inequality (13.2) gives that $\{|x|\} > \{|y|\}$. Hence, by identity $\{|x|\} = -(\{x\} - 1)$ we have that $\{x\} < \{y\}$. So, $x < y < 0$ is obtained.

Collecting the cases we have

$$(13.3) \quad x < y.$$

Sufficiency. Let us assume that $x < y$. Considering the monotonicity of the function $f(x) = (\text{sgn } x) \frac{[|x|]}{[|x|] + 1}$, we have

$$(13.4) \quad (\text{sgn } x) \frac{[|x|]}{[|x|] + 1} < (\text{sgn } y) \frac{[|y|]}{[|y|] + 1}$$

or

$$(13.5) \quad (\text{sgn } x) \frac{[|x|]}{[|x|] + 1} = (\text{sgn } y) \frac{[|y|]}{[|y|] + 1}.$$

In case of (13.4), Definition 11 and (1.1) show that $\overset{\square}{x} < \overset{\square}{y}$.

In case of (13.5) the cases $|x| \geq 1$ and $|y| < 1$; $|x| < 1$ and $|y| \geq 1$ are not allowed. So, we have the following two cases

$$a) \quad 0 \leq |x|, |y| < 1$$

or

$$b) \quad |x|, |y| \geq 1,$$

only.

In the case a) if $x = 0$ then $y > 0$, if $y = 0$ then $x < 0$. Otherwise, $0 < |x|, |y| < 1$. Clearly, $[|x|] = [|y|] = 0$, so $\{|x|\} = |x|$, $\{|y|\} = |y|$. The inequality $x < y$ implies that $\text{sgn } x \leq \text{sgn } y$.
If $\text{sgn } x < \text{sgn } y$ then $-1 < x < 0 < y < 1$. So,

$$(\text{sgn } x) \text{ area th}\{|x|\} < 0 < (\text{sgn } x) \text{ area th}\{|y|\}$$

and Definition 11 by (1.1) gives that $\overline{x} < \overline{y}$.

If $\text{sgn } x = \text{sgn } y = 1$ then $0 < x < y < 1$. So,

$$0 < (\text{sgn } x) \text{ area th}\{|x|\} < (\text{sgn } y) \text{ area th}\{|y|\}$$

and Definition 11 by (1.1) gives that $\overline{x} < \overline{y}$.

If $\text{sgn } x = \text{sgn } y = -1$, then $-1 < x < y < 0$. Hence, $0 < |y| < |x| < 1$ and $0 < \{|y|\} < \{|x|\} < 1$. So

$$0 > (\text{sgn } y) \text{ area th}\{|y|\} > (\text{sgn } x) \text{ area th}\{|x|\} > -1$$

and Definition 11 by (1.1) gives that $\overline{x} < \overline{y}$.

In the case b) (13.5) yields

$$[|x|] = [|y|].$$

Integer x and y are not allowed.

If $\text{sgn } x = \text{sgn } y = 1$, then the identity $\{|x|\} = x - [|x|]$ by $x < y$ implies that $\{|x|\} < \{|y|\}$. Hence, $(\text{sgn } x) \text{ area th}\{|x|\} < (\text{sgn } y) \text{ area th}\{|y|\}$. So, Definition 11 by (1.1) gives that $\overline{x} < \overline{y}$.

The case $\text{sgn } x = 1$ and $\text{sgn } y = -1$ is not allowed.

If $\text{sgn } x = -1$ and $\text{sgn } y = 1$ then $(\text{sgn } x) \text{ area th}\{|x|\} < (\text{sgn } y) \text{ area th}\{|y|\}$. So Definition 11 by (1.1) gives that $\overline{x} < \overline{y}$.

If $\text{sgn } x = \text{sgn } y = -1$, then identity $\{|x|\} = -x - [|x|]$. So, inequality $x < y$ implies $-x > -y > 1$. Hence, $\{|x|\} > \{|y|\}$. So, $(\text{sgn } x) \text{ area th}\{|x|\} < (\text{sgn } y) \text{ area th}\{|y|\}$ and Definition 11 by (1.1) gives that $\overline{x} < \overline{y}$.

Remark. Theorem 13 shows that the Postulate of ordering is fulfilled.

$$\overline{R}$$

Theorem 14. If $x, y \in \mathbf{R}$ then $x < y \iff x < y$.

Proof. Identity (7.2) and Theorem 13 show that $x < y \iff \overline{x} < \overline{y}$. By (7.1) we have that $\overline{x} = th \ x$ and $\overline{y} = th \ y$. Using the strict monotonicity of the function th we have that $\overline{x} < \overline{y} \iff x < y$.

Remark. Theorem 14 shows that the Requirement of ordering is fulfilled.

$$\overline{R}$$

Remark 15. By Theorem 14 we may use $u < v$ instead of $u < v$ for any $u, v \in \overline{R}$. Theorem 13 with identity (7.2) gives

Theorem 16. (*Monotonicity of compression*) For any $u, v \in \overline{R}$ the inequality $\overline{u} < \overline{v}$ holds if and only if $u < v$. Moreover, Theorem 13 yields the following corollaries:

Corollary 17. The relation " $<$ " is irreflexive, anti-symmetrical and transitive.

Corollary 18. (*Trichotomy*) For any $x, y \in \mathbf{R}$ from among relations $\overline{x} < \overline{y}$, $\overline{x} = \overline{y}$ and $\overline{x} > \overline{y}$ one and only one is true.

Definition 19. (*Super-addition*) For any $x, y \in \mathbf{R}$, the super-sum of $\overline{x}$ and $\overline{y}$ is

$$(19.1) \quad \overline{x} \bigoplus \overline{y} = (\text{sgn}(x+y)) \left(\text{area th}\{|x+y|\} + \frac{i}{2} \frac{[|x+y|]}{[|x+y|]+1} \right).$$

By Definition 1 the identity

$$(19.2) \quad \overline{x} \text{---}\bigoplus\text{---}\overline{y} = \overline{x+y}, \quad x, y \in \mathbf{R} \quad (\text{See Postulate of super-addition})$$

is obvious.

Definition 20. (*Super-multiplication*) For any $x, y \in \mathbf{R}$, the super-multiplication of $\overline{x}$ and $\overline{y}$ is

$$(20.1) \quad \overline{x} \text{---}\bigodot\text{---}\overline{y} = (\text{sgn}(x \cdot y))(\text{area th}\{|x \cdot y|\}) + \frac{i}{2} \frac{[|x \cdot y|]}{[|x \cdot y|] + 1}.$$

By Definition 1 the identity

$$(20.2) \quad \overline{x} \text{---}\bigodot\text{---}\overline{y} = \overline{x \cdot y}, \quad x, y \in \mathbf{R} \quad (\text{See Postulate of super-multiplication})$$

is obvious.

Remark 21. Using identities (19.2) and (20.2) we find that the field $(\mathbf{R}, +, \cdot)$ is isomorphic with the algebraic structure $(\overline{R}, \text{---}\bigoplus\text{---}, \text{---}\bigodot\text{---})$; so the latter is also a field with the operations super-addition and super-multiplication. By (19.1) we can see that the additive unit element of $\overline{R}$ is $\overline{0} = 0$. The additive inverse element of $\overline{x}$ is $\overline{-x}$ for which, by (1.1), the identity

$$(21.1) \quad \overline{-x} = -\overline{x}, \quad x \in \mathbf{R}$$

holds. By (20.1) we can see that the multiplicative unit element of $\overline{R}$ is $\overline{1} = \frac{i}{4}$. The multiplicative inverse element of $\overline{x} \neq 0$ is $\overline{(\frac{1}{x})}$.

Remark 22. By (7.1) we have that for any $u \in \overline{R}$ the identity

$$(22.1) \quad \overline{-u} = -\overline{u}, \quad u \in \overline{R}$$

holds. Moreover, denoting $\overline{x} = u$ and $\overline{y} = v$, the identities (19.2) and (20.2) by (9.1) yield the identities

$$(22.2) \quad u \text{---}\bigoplus\text{---}v = \overline{\overline{u} + \overline{v}} \quad (u, v \in \overline{R})$$

and

$$(22.3) \quad u \text{---}\bigodot\text{---}v = \overline{\overline{u} \cdot \overline{v}} \quad (u, v \in \overline{R}),$$

respectively.

Definition 23. The exploded real number u is called positive if $u > 0$ and negative if $u < 0$. (These are extensions of the familiar positivity and negativity of real numbers.)

Theorem 24. (*Monotonicity of super-addition*) Let u, v and w be arbitrary exploded real numbers. If $u < v$ then

$$u \text{---}\bigoplus\text{---}w < v \text{---}\bigoplus\text{---}w.$$

Proof. Using (22.2), Theorem 16, Theorem 13 and (22.2) again, we have that

$$u \text{---}\bigoplus\text{---}w = \overline{\overline{u} + \overline{w}} < \overline{\overline{v} + \overline{w}} = v \text{---}\bigoplus\text{---}w.$$

Theorem 25. (*Monotony of super-multiplication*) Let u, v be arbitrary and w positive exploded real numbers. If $u < v$ then $u \text{---} \bigcirc \text{---} w < v \text{---} \bigcirc \text{---} w$.

Proof. First, we mention that by Theorem 16 and Definition 23 with Definition (7.1) $\frac{w}{\square} > \frac{0}{\square} = 0$ is obtained. Moreover, using (22.3), Theorem 16, Theorem 13 and (22.3) again, we have that

$$u \text{---} \bigcirc \text{---} w = \overline{\frac{u}{\square} \cdot \frac{w}{\square}} < \overline{\frac{v}{\square} \cdot \frac{w}{\square}} = v \text{---} \bigcirc \text{---} w.$$

Remark 26. Considering Remark 21, Theorem 24 and Theorem 25 we can see that $(\frac{\square}{R}, \text{---} \bigoplus \text{---}, \text{---} \bigcirc \text{---})$ is an ordered field.

Reference

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Received October 27, 2003.

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