

COEFFICIENT PROBLEMS FOR A CLASS OF ANALYTIC FUNCTIONS INVOLVING HADAMARD PRODUCTS

MASLINA DARUS

ABSTRACT. For $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$ and $\beta > 0$, let $MS(\Phi, \Psi; \lambda, \alpha, \beta)$ be the class of functions defined in the open unit disc D by

$$\left| \arg \left(\frac{\lambda z^2 f''(z) + z f'(z)}{\lambda z g'(z) + (1 - \lambda)g(z)} \right) \right| < \frac{\pi \alpha}{2}, \quad z \in D$$

where $g(z) = z + b_2 z^2 + b_3 z^3 + \dots$ is analytic function and satisfies

$$\left| \arg \left(\frac{g(z) * \Phi(z)}{g(z) * \Psi(z)} \right) \right| < \frac{\pi \beta}{2}, \quad z \in D$$

for some $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in D such that $g(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ ($n \geq 2$). For $f \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$ and given by $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$, a sharp upper bound is obtained for $|a_3 - \mu a_2^2|$ when $\mu \geq 1$.

1. INTRODUCTION

Let A denote the family of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are in the open unit disk $D = \{z : |z| < 1\}$. Further, let S denote the class of functions which are univalent in D .

Let the function $f(z)$ be defined by (1.1). Also let the function $g(z)$ be defined by

$$(1.2) \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$ is defined by

$$(1.3) \quad f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic functions, normalised, functions, univalent functions.

The work presented here was supported by IRPA grant 09-02-02-0080-EA208, Malaysia.

Fekete and Szegő [8] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ when μ is real. For various subclasses of S , sharp upper bound for functional $|a_3 - \mu a_2^2|$ has been studied by many different authors including [1–7, 9–11, 13–17, 19–20].

In this paper we obtain sharp upper bounds for $|a_3 - \mu a_2^2|$ when f belonging to the class of functions defined as follows:

Definition 1.1. Let $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$ and $\beta > 0$, and let $f \in A$. Then $f \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$ if and only if

$$(1.4) \quad \left| \arg \left(\frac{\lambda z^2 f''(z) + z f'(z)}{\lambda z g'(z) + (1 - \lambda)g(z)} \right) \right| < \frac{\pi \alpha}{2},$$

with $g \in A$ and satisfies

$$(1.5) \quad \left| \arg \left(\frac{g(z) * \Phi(z)}{g(z) * \Psi(z)} \right) \right| < \frac{\pi \beta}{2}, \quad z \in D$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in D such that $g(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ ($n \geq 2$).

Note that $MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1, \beta) = K(\beta)$ the class of close-to-convex functions defined in [3] and $MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1, 1) = K(1)$ is the class of normalised close-to-convex functions defined by Kaplan [12]. Whereas,

$$MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, \alpha, \beta) = K(\alpha, \beta)$$

is the class of normalised close-to-convex functions defined in [7].

2. MAIN RESULT

In order to derive our main results, we have to recall here the following lemma.

Lemma 2.1 ([18]). Let $h \in P$ that is, h be analytic in D and be given by $h(z) = 1 + c_1 z + c_2 z^2 + \dots$, and $\operatorname{Re} h(z) > 0$ for $z \in D$, then

$$(2.1) \quad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

Theorem 2.2. Let $f(z)$ be given by (1.1). If $f(z) \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$; $0 < \alpha \leq 1$, $\beta \geq 1$, $0 \leq \lambda \leq 1$ and $3\eta\mu \geq 2\delta^2 + 4\delta\gamma_2$ where $\delta = \lambda_2 - \gamma_2$, $\eta = \lambda_3 - \gamma_3$ and $\mu \geq 1$, then we have the sharp inequality

$$(2.2) \quad |a_3 - \mu a_2^2| \leq \frac{\beta^2}{3\eta\delta^2} [3\mu\eta - 2\delta(\delta + 2\gamma_2)] + \frac{\alpha(\alpha\delta + 2\beta(1 + \lambda))(3\mu(1 + 2\lambda) - 2(1 + \lambda)^2)}{3\delta(1 + \lambda)^2(1 + 2\lambda)}.$$

Proof. Let $f(z) \in MS(\Phi, \Psi; \lambda, \alpha, \beta)$. It follows from (1.4) that

$$(2.3) \quad \lambda z^2 f''(z) + z f'(z) = q^\alpha(z) [\lambda z g'(z) + (1 - \lambda)g(z)],$$

for $z \in D$, with $q \in P$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \dots$. Equating coefficients, we obtain

$$(2.4) \quad 2a_2(1 + \lambda) = b_2(1 + \lambda) + \alpha q_1$$

and

$$(2.5) \quad 3(1+2\lambda)a_3 = \alpha q_2 + \alpha q_1 b_2(1+\lambda) + \frac{\alpha(\alpha-1)}{2}q_1^2 + b_3(1+2\lambda).$$

Also, it follows from (1.5) that

$$(2.6) \quad g(z) * \Phi(z) = (g(z) * \Psi(z))p^\beta(z),$$

where $p \in P$ with $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$ for $z \in D$. Thus equating coefficients, we obtain

$$(2.7) \quad \delta b_2 = \beta p_1$$

and

$$(2.8) \quad \eta b_3 = \beta \left(p_2 + \frac{\beta(\delta + 2\gamma_2) - \delta}{2\delta} p_1^2 \right).$$

From (2.4), (2.5), (2.7) and (2.8) we have

$$(2.9) \quad \begin{aligned} a_3 - \mu a_2^2 &= \frac{\alpha}{3(1+2\lambda)} \left(q_2 - \frac{q_1^2}{2} \right) + \frac{\beta}{3\eta} \left(p_2 - \frac{p_1^2}{2} \right) \\ &+ \frac{\alpha^2 q_1^2 \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}}{12(1+\lambda)^2(1+2\lambda)} \\ &+ \frac{\alpha\beta p_1 q_1 \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}}{6\delta(1+\lambda)(1+2\lambda)} \\ &+ \frac{\beta^2 p_1^2 \{2\delta^2 + 4\gamma_2\delta - 3\mu\eta\}}{12\eta\delta^2}. \end{aligned}$$

□

Assume that $a_3 - \mu a_2^2$ positive. Thus we now estimate $\operatorname{Re}(a_3 - \mu a_2^2)$ by applying the same technique done by London [16]. And so from (2.9) and by using lemma 2.1 and letting $p_1 = 2re^{i\theta}$, $q_1 = 2Re^{i\phi}$, $0 \leq r \leq 1$, $0 \leq R \leq 1$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq 2\pi$, we obtain

$$\begin{aligned} \operatorname{Re}(a_3 - \mu a_2^2) &= \frac{\alpha}{3(1+2\lambda)} \operatorname{Re} \left(q_2 - \frac{q_1^2}{2} \right) + \frac{\alpha^2 \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\} \operatorname{Re} q_1^2}{12(1+\lambda)^2(1+2\lambda)} \\ &+ \frac{\beta}{3\eta} \operatorname{Re} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{\alpha\beta \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\} \operatorname{Re} p_1 q_1}{6\delta(1+\lambda)(1+2\lambda)} \\ &+ \frac{\beta^2 \{2\delta^2 + 4\gamma_2\delta - 3\mu\eta\} \operatorname{Re} p_1^2}{12\eta\delta^2} \\ &\leq \frac{2\alpha}{3(1+2\lambda)}(1-R^2) + \frac{\alpha^2 \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\} R^2 \cos 2\phi}{3(1+\lambda)^2(1+2\lambda)} \\ &+ \frac{2\beta}{3\eta}(1-r^2) + \frac{2\alpha\beta \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\} rR \cos(\theta + \phi)}{3\delta(1+\lambda)(1+2\lambda)} \\ &+ \frac{\beta^2 \{2\delta^2 + 4\gamma_2\delta - 3\mu\eta\} r^2 \cos 2\theta}{3\eta\delta^2} \\ &\leq \frac{2\alpha}{3(1+2\lambda)}(1-R^2) + \frac{\alpha^2 \{3\mu(1+2\lambda) - 2(1+\lambda)^2\} R^2}{3(1+\lambda)^2(1+2\lambda)} \\ &+ \frac{2\beta}{3\eta}(1-r^2) + \frac{2\alpha\beta \{3\mu(1+2\lambda) - 2(1+\lambda)^2\} rR}{3\delta(1+\lambda)(1+2\lambda)} \end{aligned}$$

$$+ \frac{\beta^2 \{3\mu\eta - (2\delta^2 + 4\gamma_2\delta)\}r^2}{3\eta\delta^2} = G(r, R).$$

Letting α , β and μ fixed and differentiating $G(r, R)$ partially when $0 < \alpha \leq 1$, $\beta \geq 1$, and $\mu \geq 1$ we observe that

$$\begin{aligned} & G_{rr}G_{RR} - (G_{rR})^2 \\ &= \frac{16\alpha\beta}{9\eta(1+2\lambda)} + \frac{4\alpha^2\beta^2(3\mu\eta - (2\delta^2 + 4\gamma_2\delta))(3\mu(1+2\lambda) - 2(1+\lambda)^2)}{9\delta^2\eta(1+\lambda)^2(1+2\lambda)} \\ &\quad - \frac{8\alpha^2\beta(3\mu(1+2\lambda) - 2(1+\lambda)^2)}{9\eta(1+\lambda)^2(1+2\lambda)} - \frac{8\alpha\beta^2(3\mu\eta - (2\delta^2 + 4\gamma_2\delta))}{9\delta^2\eta(1+2\lambda)} \\ &\quad - \frac{4\alpha^2\beta^2(3\mu(1+2\lambda) - 2(1+\lambda)^2)^2}{9\delta^2(1+\lambda)^2(1+2\lambda)^2} < 0. \end{aligned}$$

Therefore, the maximum of $G(r, R)$ occurs on the boundaries. Thus the desired inequality follows by observing that $G(r, R) \leq G(1, 1) = \frac{\beta^2}{3\eta\delta^2}[3\mu\eta - 2\delta(\delta + 2\gamma_2)]$

$$(2.10) \quad + \frac{\alpha(\alpha\delta + 2\beta(1+\lambda))(3\mu(1+2\lambda) - 2(1+\lambda)^2)}{3\delta(1+\lambda)^2(1+2\lambda)}.$$

The equality is attained when choosing $p_1 = q_1 = 2i$ and $p_2 = q_2 = -2$ in (2.9).

Remark 1. Letting $\lambda = 0$ in Theorem 2.2, we have the result given by Darus and Thomas [7].

Remark 2. Letting $\Phi(z) = \frac{z}{(1-z)^2}$, $\Upsilon(z) = \frac{z}{1-z}$, $\lambda = 0$ and $\alpha = 1$ in Theorem 2.2, we have the result given by Jahangiri [11].

REFERENCES

- [1] H. R. Abdel-Gawad and D. K. Thomas. A subclass of close-to-convex functions. *Publ. Inst. Math. (Beograd) (N.S.)*, 49(63):61–66, 1991.
- [2] H. R. Abdel-Gawad and D. K. Thomas. The Fekete-Szegő problem for strongly close-to-convex functions. *Proc. Amer. Math. Soc.*, 114(2):345–349, 1992.
- [3] A. Chonweerayoot, D. K. Thomas, and W. Upakarnitikaset. On the coefficients of close-to-convex functions. *Math. Japon.*, 36(5):819–826, 1991.
- [4] M. Darus. On the coefficient problems with Hadamard product. *J. Anal. Appl.*, 2(2):87–93, 2004.
- [5] M. Darus and D. K. Thomas. On the Fekete-Szegő theorem for close-to-convex functions. *Math. Japon.*, 44(3):507–511, 1996.
- [6] M. Darus and D. K. Thomas. On the Fekete-Szegő theorem for close-to-convex functions. *Math. Japon.*, 47(1):125–132, 1998.
- [7] M. Darus and D. K. Thomas. The Fekete-Szegő theorem for strongly close-to-convex functions. *Sci. Math.*, 3(2):201–212 (electronic), 2000.
- [8] M. Fekete and G. Szegő. Eine Bemerkung ber ungerade schlichte Funktionen. *J. Lond. Math. Soc.*, 8:85–89, 1933.
- [9] B. Frasin and M. Darus. On the Fekete-Szegő problem using Hadamard product. *Int. Math. J.*, 3(12):1289–1295, 2003.
- [10] R. M. Goel and B. S. Mehrotra. A coefficient inequality for certain classes of analytic functions. *Tamkang J. Math.*, 22(2):153–163, 1991.
- [11] M. Jahangiri. A coefficient inequality for a class of close-to-convex functions. *Math. Japon.*, 41(3):557–559, 1995.
- [12] W. Kaplan. Close-to-convex schlicht functions. *Michigan Math. J.*, 1:169–185 (1953), 1952.

- [13] F. R. Keogh and E. P. Merkes. A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.*, 20:8–12, 1969.
- [14] W. Koepf. On the Fekete-Szegő problem for close-to-convex functions. *Proc. Amer. Math. Soc.*, 101(1):89–95, 1987.
- [15] W. Koepf. On the Fekete-Szegő problem for close-to-convex functions. II. *Arch. Math. (Basel)*, 49(5):420–433, 1987.
- [16] R. R. London. Fekete-Szegő inequalities for close-to-convex functions. *Proc. Amer. Math. Soc.*, 117(4):947–950, 1993.
- [17] M. A. Nasr and H. R. El-Gawad. On the Fekete-Szegő problem for close-to-convex functions of order ρ . In *New trends in geometric function theory and applications (Madras, 1990)*, pages 66–74. World Sci. Publishing, River Edge, NJ, 1991.
- [18] C. Pommerenke. *Univalent functions*. Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen, *Studia Mathematica/Mathematische Lehrbücher*, Band XXV.
- [19] H. M. Srivastava, A. K. Mishra, and M. K. Das. The Fekete-Szegő problem for a subclass of close-to-convex functions. *Complex Variables Theory Appl.*, 44(2):145–163, 2001.
- [20] S. Y. Trimble. A coefficient inequality for convex univalent functions. *Proc. Amer. Math. Soc.*, 48:266–267, 1975.

Received March 03, 2004.

SCHOOL OF MATHEMATICAL SCIENCES,
FACULTY OF SCIENCES AND TECHNOLOGY,
UNIVERSITI KEBANGSAAN MALAYSIA,
BANGI 43600 SELANGOR, MALAYSIA
E-mail address: maslina@pkrisc.cc.ukm.my