

THE BECKMAN-QUARLES THEOREM FOR MAPPINGS FROM $\mathbb{C}^2$ TO $\mathbb{C}^2$

APOLONIUSZ TYSZKA

ABSTRACT. Let $\varphi: \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}$, $\varphi((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2$. We say that $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ preserves distance $d \geq 0$, if for each $X, Y \in \mathbb{C}^2$ $\varphi(X, Y) = d^2$ implies $\varphi(f(X), f(Y)) = d^2$. We prove that each unit-distance preserving mapping $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ has a form $I \circ (\gamma, \gamma)$, where $\gamma: \mathbb{C} \rightarrow \mathbb{C}$ is a field homomorphism and $I: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is an affine mapping with orthogonal linear part. We prove an analogous result for mappings from $\mathbf{K}^2$ to $\mathbf{K}^2$, where $\mathbf{K}$ is a field such that $\text{char}(\mathbf{K}) \notin \{2, 3, 5\}$ and -1 is a square.

The classical Beckman-Quarles theorem states that each unit-distance preserving mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ ($n \geq 2$) is an isometry, see [1]–[5]. Let $\varphi: \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}$, $\varphi((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2$. We say that $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ preserves distance $d \geq 0$, if for each $X, Y \in \mathbb{C}^2$ $\varphi(X, Y) = d^2$ implies $\varphi(f(X), f(Y)) = d^2$. If $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ and for each $X, Y \in \mathbb{C}^2$ $\varphi(X, Y) = \varphi(f(X), f(Y))$, then f is an affine mapping with orthogonal linear part; it follows from a general theorem proved in [3, 58 ff], see also [4, p. 30]. The author proved in [9]: each unit-distance preserving mapping $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ satisfies

$$(1) \quad \varphi(X, Y) = \varphi(f(X), f(Y))$$

for all $X, Y \in \mathbb{C}^2$ with rational $\varphi(X, Y)$.

Theorem 1. *If $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ preserves unit distance, $f((0, 0)) = (0, 0)$, $f((1, 0)) = (1, 0)$ and $f((0, 1)) = (0, 1)$, then there exists a field homomorphism $\rho: \mathbb{R} \rightarrow \mathbb{C}$ satisfying $\forall x_1, x_2 \in \mathbb{C}$:*

$$(2) \quad \begin{aligned} f((x_1, x_2)) \in \{ & (\rho(\text{Re}(x_1)) + \rho(\text{Im}(x_1)) \cdot \mathbf{i}, \rho(\text{Re}(x_2)) + \rho(\text{Im}(x_2)) \cdot \mathbf{i}), \\ & (\rho(\text{Re}(x_1)) - \rho(\text{Im}(x_1)) \cdot \mathbf{i}, \rho(\text{Re}(x_2)) - \rho(\text{Im}(x_2)) \cdot \mathbf{i}) \}. \end{aligned}$$

Proof. Obviously, $g = f|_{\mathbb{R}^2}: \mathbb{R}^2 \rightarrow \mathbb{C}^2$ preserves unit distance. The author proved in [8] that such a g has a form $I \circ (\rho, \rho)$, where $\rho: \mathbb{R} \rightarrow \mathbb{C}$ is a field homomorphism and $I: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is an affine mapping with orthogonal linear part. Since $f((0, 0)) = (0, 0)$, $f((1, 0)) = (1, 0)$, $f((0, 1)) = (0, 1)$, we conclude that $f|_{\mathbb{R}^2} = (\rho, \rho)$. From this, condition (2) holds true if $(x_1, x_2) \in \mathbb{R}^2$. Assume now that $(x_1, x_2) \in \mathbb{C}^2 \setminus \mathbb{R}^2$.

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Let $x_1 = a_1 + b_1 \cdot \mathbf{i}$, $x_2 = a_2 + b_2 \cdot \mathbf{i}$, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$, and, for example $b_1 \neq 0$. For each $t \in \mathbb{R}$

$$\varphi((a_1 + b_1 \cdot \mathbf{i}, a_2 + b_2 \cdot \mathbf{i}), (a_1 + tb_2, a_2 - tb_1)) = (t^2 - 1)(b_1^2 + b_2^2).$$

By this and (1): for each $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ we have:

$$(3) \quad \varphi(f((a_1 + b_1 \cdot \mathbf{i}, a_2 + b_2 \cdot \mathbf{i})), f((a_1 + tb_2, a_2 - tb_1))) = (t^2 - 1)(b_1^2 + b_2^2).$$

Let $f((a_1 + b_1 \cdot \mathbf{i}, a_2 + b_2 \cdot \mathbf{i})) = (y_1, y_2)$. From (3) and $f|_{\mathbb{R}^2} = (\rho, \rho)$ we obtain: for each $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ we have:

$$(4) \quad (y_1 - \rho(a_1) - \rho(t)\rho(b_2))^2 + (y_2 - \rho(a_2) + \rho(t)\rho(b_1))^2 = (t^2 - 1)(b_1^2 + b_2^2).$$

For each $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ we have:

$$(t^2 - 1)(b_1^2 + b_2^2) = \rho((t^2 - 1)(b_1^2 + b_2^2)) = (\rho(t)^2 - 1)(\rho(b_1)^2 + \rho(b_2)^2).$$

By this and (4): for each $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ we have:

$$(5) \quad \begin{aligned} & (y_1 - \rho(a_1))^2 + (y_2 - \rho(a_2))^2 + \rho(b_1)^2 + \rho(b_2)^2 + 2\rho(t) \\ & \times (\rho(b_1)(y_2 - \rho(a_2)) - \rho(b_2)(y_1 - \rho(a_1))) = 0. \end{aligned}$$

There are infinitely many $t \in \mathbb{R}$ with rational $(t^2 - 1)(b_1^2 + b_2^2)$ and ρ is injective. From these two facts and (5), we obtain:

$$(6) \quad \rho(b_1)(y_2 - \rho(a_2)) - \rho(b_2)(y_1 - \rho(a_1)) = 0$$

and

$$(7) \quad (y_1 - \rho(a_1))^2 + (y_2 - \rho(a_2))^2 + \rho(b_1)^2 + \rho(b_2)^2 = 0.$$

By (6):

$$(8) \quad y_2 - \rho(a_2) = \frac{\rho(b_2)}{\rho(b_1)} \cdot (y_1 - \rho(a_1)).$$

Applying (8) to (7) we get:

$$(y_1 - \rho(a_1))^2 + \frac{\rho(b_2)^2}{\rho(b_1)^2} \cdot (y_1 - \rho(a_1))^2 + \rho(b_1)^2 + \rho(b_2)^2 = 0.$$

It gives $\left(\frac{(y_1 - \rho(a_1))^2}{\rho(b_1)^2} + 1 \right) \cdot (\rho(b_1)^2 + \rho(b_2)^2) = 0$. Since $\rho(b_1)^2 + \rho(b_2)^2 \neq 0$, we get

$$\underbrace{y_1 = \rho(a_1) + \rho(b_1) \cdot \mathbf{i}}_{\text{case 1}} \text{ or } \underbrace{y_1 = \rho(a_1) - \rho(b_1) \cdot \mathbf{i}}_{\text{case 2}}.$$

In case 1, by (8)

$$\begin{aligned} y_2 &= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (y_1 - \rho(a_1)) \\ &= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (\rho(a_1) + \rho(b_1) \cdot \mathbf{i} - \rho(a_1)) \\ &= \rho(a_2) + \rho(b_2) \cdot \mathbf{i}. \end{aligned}$$

In case 2, by (8)

$$\begin{aligned} y_2 &= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (y_1 - \rho(a_1)) \\ &= \rho(a_2) + \frac{\rho(b_2)}{\rho(b_1)} \cdot (\rho(a_1) - \rho(b_1) \cdot \mathbf{i} - \rho(a_1)) \\ &= \rho(a_2) - \rho(b_2) \cdot \mathbf{i}. \end{aligned}$$

The proof is completed. $\square$

Let $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ preserves unit distance, $f((0, 0)) = (0, 0)$, $f((1, 0)) = (1, 0)$ and $f((0, 1)) = (0, 1)$. Theorem 1 provides a field homomorphism $\rho: \mathbb{R} \rightarrow \mathbb{C}$ satisfying (2). By Theorem 1 the sets

$$\mathbf{A} = \{(x_1, x_2) \in \mathbb{C}^2 : f((x_1, x_2)) = (\rho(\operatorname{Re}(x_1)) + \rho(\operatorname{Im}(x_1)) \cdot \mathbf{i}, \rho(\operatorname{Re}(x_2)) + \rho(\operatorname{Im}(x_2)) \cdot \mathbf{i})\}$$

and

$$\mathbf{B} = \{(x_1, x_2) \in \mathbb{C}^2 : f((x_1, x_2)) = (\rho(\operatorname{Re}(x_1)) - \rho(\operatorname{Im}(x_1)) \cdot \mathbf{i}, \rho(\operatorname{Re}(x_2)) - \rho(\operatorname{Im}(x_2)) \cdot \mathbf{i})\}$$

satisfy $\mathbf{A} \cup \mathbf{B} = \mathbb{C}^2$. The mapping

$$\mathbb{C} \ni x \xrightarrow{\theta} \rho(\operatorname{Re}(x)) + \rho(\operatorname{Im}(x)) \cdot \mathbf{i} \in \mathbb{C}$$

is a field homomorphism, θ extends ρ ,

$$\mathbf{A} = \{(x_1, x_2) \in \mathbb{C}^2 : f((x_1, x_2)) = (\theta(x_1), \theta(x_2))\}.$$

The mapping

$$\mathbb{C} \ni x \xrightarrow{\zeta} \rho(\operatorname{Re}(x)) - \rho(\operatorname{Im}(x)) \cdot \mathbf{i} \in \mathbb{C}$$

is a field homomorphism, ζ extends ρ ,

$$\mathbf{B} = \{(x_1, x_2) \in \mathbb{C}^2 : f((x_1, x_2)) = (\zeta(x_1), \zeta(x_2))\}.$$

We would like to prove $f = (\theta, \theta)$ or $f = (\zeta, \zeta)$; we will prove it later in Theorem 2.

Let $\psi: \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{R}$, $\psi((x_1, x_2), (y_1, y_2)) = \operatorname{Im}(x_1) \cdot \operatorname{Im}(y_1) + \operatorname{Im}(x_2) \cdot \operatorname{Im}(y_2)$.

Lemma 1. *If $x_1, x_2, y_1, y_2 \in \mathbb{C}$, $\varphi((x_1, x_2), (y_1, y_2)) \in \mathbb{Q}$ and $\psi((x_1, x_2), (y_1, y_2)) \neq 0$, then*

$$(9) \quad (y_1, y_2) \in \mathbf{A} \text{ implies } (x_1, x_2) \in \mathbf{A}$$

and

$$(10) \quad (y_1, y_2) \in \mathbf{B} \text{ implies } (x_1, x_2) \in \mathbf{B}.$$

Proof. We prove only (9), the proof of (10) follows analogically.

Let $\varphi((x_1, x_2), (y_1, y_2)) = r \in \mathbb{Q}$. Assume, on the contrary, that $(y_1, y_2) \in \mathbf{A}$ and $(x_1, x_2) \notin \mathbf{A}$. Since $\mathbf{A} \cup \mathbf{B} = \mathbb{C}^2$, $(x_1, x_2) \in \mathbf{B}$. Let $x_1 = a_1 + b_1 \cdot \mathbf{i}$, $x_2 = a_2 + b_2 \cdot \mathbf{i}$, $y_1 = \tilde{a}_1 + \tilde{b}_1 \cdot \mathbf{i}$, $y_2 = \tilde{a}_2 + \tilde{b}_2 \cdot \mathbf{i}$, where $a_1, b_1, a_2, b_2, \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2 \in \mathbb{R}$. By (1):

$$\begin{aligned} (11) \quad r &= \varphi((x_1, x_2), (y_1, y_2)) = \varphi(f((x_1, x_2)), f((y_1, y_2))) \\ &= (\rho(a_1) - \rho(b_1) \cdot \mathbf{i} - \rho(\tilde{a}_1) - \rho(\tilde{b}_1) \cdot \mathbf{i})^2 \\ &\quad + (\rho(a_2) - \rho(b_2) \cdot \mathbf{i} - \rho(\tilde{a}_2) - \rho(\tilde{b}_2) \cdot \mathbf{i})^2. \end{aligned}$$

Since $r \in \mathbb{Q}$,

$$\begin{aligned}
 r &= \theta(r) = \theta((a_1 + b_1 \cdot \mathbf{i} - \tilde{a}_1 - \tilde{b}_1 \cdot \mathbf{i})^2 + (a_2 + b_2 \cdot \mathbf{i} - \tilde{a}_2 - \tilde{b}_2 \cdot \mathbf{i})^2) \\
 (12) \quad &= (\rho(a_1) + \rho(b_1) \cdot \mathbf{i} - \rho(\tilde{a}_1) - \rho(\tilde{b}_1) \cdot \mathbf{i})^2 \\
 &\quad + (\rho(a_2) + \rho(b_2) \cdot \mathbf{i} - \rho(\tilde{a}_2) - \rho(\tilde{b}_2) \cdot \mathbf{i})^2.
 \end{aligned}$$

Subtracting (11) and (12) by sides we obtain:

$$2\rho(b_1) \cdot \mathbf{i} \cdot (2\rho(\tilde{b}_1) \cdot \mathbf{i} - 2\rho(a_1) + 2\rho(\tilde{a}_1)) + 2\rho(b_2) \cdot \mathbf{i} \cdot (2\rho(\tilde{b}_2) \cdot \mathbf{i} - 2\rho(a_2) + 2\rho(\tilde{a}_2)) = 0.$$

Thus

$$(13) \quad -\rho(b_1 \tilde{b}_1 + b_2 \tilde{b}_2) = \rho(b_1(a_1 - \tilde{a}_1) + b_2(a_2 - \tilde{a}_2)) \cdot \mathbf{i}.$$

Squaring both sides of (13) we get:

$$\rho((b_1 \tilde{b}_1 + b_2 \tilde{b}_2)^2 + (b_1(a_1 - \tilde{a}_1) + b_2(a_2 - \tilde{a}_2))^2) = 0,$$

so in particular $\psi((x_1, x_2), (y_1, y_2)) = b_1 \tilde{b}_1 + b_2 \tilde{b}_2 = 0$, a contradiction. $\square$

The next lemma is obvious.

Lemma 2. *For each $S, T \in \mathbb{R}^2$ there exist $n \in \{1, 2, 3, \dots\}$ and $P_1, \dots, P_n \in \mathbb{R}^2$ such that $\|S - P_1\| = \|P_1 - P_2\| = \dots = \|P_{n-1} - P_n\| = \|P_n - T\| = 1$.*

Lemma 3. *For each $X \in \mathbb{C}^2 \setminus \mathbb{R}^2$*

$$(\mathbf{i}, \mathbf{i}) \in \mathbf{A} \text{ implies } X \in \mathbf{A}$$

and

$$(\mathbf{i}, \mathbf{i}) \in \mathbf{B} \text{ implies } X \in \mathbf{B}.$$

Proof. Let $X = (a_1 + b_1 \cdot \mathbf{i}, a_2 + b_2 \cdot \mathbf{i})$, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$. Since $X \in \mathbb{C}^2 \setminus \mathbb{R}^2$, $b_1 \neq 0$ or $b_2 \neq 0$. Assume that $b_1 \neq 0$, when $b_2 \neq 0$ the proof is analogous. The points $S = (a_1 + \sqrt{1 + b_2^2}, a_2 + \sqrt{1 + (b_1 - 1)^2})$ and $T = (\sqrt{2}, 0)$ belong to $\mathbb{R}^2$. Applying Lemma 2 we find $P_1, \dots, P_n \in \mathbb{R}^2$ satisfying $\|S - P_1\| = \|P_1 - P_2\| = \dots = \|P_{n-1} - P_n\| = \|P_n - T\| = 1$. The points

$$X_1 = X,$$

$$X_2 = \left(a_1 + \sqrt{1 + b_2^2} + b_1 \cdot \mathbf{i}, a_2 \right),$$

$$X_3 = S + (\mathbf{i}, 0) = \left(a_1 + \sqrt{1 + b_2^2} + \mathbf{i}, a_2 + \sqrt{1 + (b_1 - 1)^2} \right),$$

$$X_4 = P_1 + (\mathbf{i}, 0),$$

$$X_5 = P_2 + (\mathbf{i}, 0),$$

...

$$X_{n+3} = P_n + (\mathbf{i}, 0),$$

$$X_{n+4} = T + (\mathbf{i}, 0) = (\sqrt{2} + \mathbf{i}, 0),$$

$$X_{n+5} = (\mathbf{i}, \mathbf{i})$$

satisfy $\varphi(X_{k-1}, X_k) = 1$ for $k \in \{2, 3, \dots, n+5\}$; $\psi(X_1, X_2) = b_1^2 \neq 0$, $\psi(X_2, X_3) = b_1 \neq 0$ and $\psi(X_{k-1}, X_k) = 1$ for $k \in \{4, 5, \dots, n+5\}$.

By Lemma 1 for each $k \in \{2, 3, \dots, n+5\}$

$$X_k \in \mathbf{A} \text{ implies } X_{k-1} \in \mathbf{A}$$

and

$$X_k \in \mathbf{B} \text{ implies } X_{k-1} \in \mathbf{B}.$$

Therefore, $(\mathbf{i}, \mathbf{i}) = X_{n+5} \in \mathbf{A}$ implies $X = X_1 \in \mathbf{A}$, and also, $(\mathbf{i}, \mathbf{i}) = X_{n+5} \in \mathbf{B}$ implies $X = X_1 \in \mathbf{B}$. $\square$

Theorem 2. *If $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ preserves unit distance, $f((0,0)) = (0,0)$, $f((1,0)) = (1,0)$ and $f((0,1)) = (0,1)$, then there exists a field homomorphism $\gamma: \mathbb{C} \rightarrow \mathbb{C}$ satisfying $f = (\gamma, \gamma)$.*

Proof. By Lemma 3

$$(\mathbf{i}, \mathbf{i}) \in \mathbf{A} \text{ implies } \mathbb{C}^2 \setminus \mathbb{R}^2 \subseteq \mathbf{A}$$

and

$$(\mathbf{i}, \mathbf{i}) \in \mathbf{B} \text{ implies } \mathbb{C}^2 \setminus \mathbb{R}^2 \subseteq \mathbf{B}.$$

Obviously, $\mathbb{R}^2 \subseteq \mathbf{A}$ and $\mathbb{R}^2 \subseteq \mathbf{B}$. Therefore,

$$\mathbf{A} = \mathbb{C}^2 \text{ and } f = (\theta, \theta), \text{ if } (\mathbf{i}, \mathbf{i}) \in \mathbf{A},$$

and also,

$$\mathbf{B} = \mathbb{C}^2 \text{ and } f = (\zeta, \zeta), \text{ if } (\mathbf{i}, \mathbf{i}) \in \mathbf{B}.$$

$\square$

As a corollary of Theorem 2 we get:

Theorem 3. *Each unit-distance preserving mapping $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ has a form $I \circ (\gamma, \gamma)$, where $\gamma: \mathbb{C} \rightarrow \mathbb{C}$ is a field homomorphism and $I: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is an affine mapping with orthogonal linear part.*

Proof. By (1):

$$\begin{aligned} 1 &= \varphi((0,0), (1,0)) = \varphi(f((0,0)), f((1,0))), \\ 1 &= \varphi((0,0), (0,1)) = \varphi(f((0,0)), f((0,1))), \\ 2 &= \varphi((1,0), (0,1)) = \varphi(f((1,0)), f((0,1))). \end{aligned}$$

By the above equalities there exists an affine mapping $J: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ with orthogonal linear part such that

$$J(f((0,0))) = (0,0), \quad J(f((1,0))) = (1,0), \quad J(f((0,1))) = (0,1).$$

By Theorem 2 there exists a field homomorphism $\gamma: \mathbb{C} \rightarrow \mathbb{C}$ satisfying $J \circ f = (\gamma, \gamma)$, so $f = J^{-1} \circ (\gamma, \gamma)$. $\square$

Obviously, Theorem 3 implies (1). The author proved in [10]:

(14) if $n \geq 2$ and a continuous $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$ preserves unit distance, then f has a form $I \circ (\rho, \dots, \rho)$, where $I: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is an affine mapping with orthogonal linear part and $\rho: \mathbb{C} \rightarrow \mathbb{C}$ is the identity or the complex conjugation.

The only continuous endomorphisms of $\mathbb{C}$ are the identity and the complex conjugation, see [6, Lemma 1, p. 356]. Therefore, Theorem 3 implies (14) restricted to $n = 2$.

Let $\mathbf{K}$ be a field, $\text{char}(\mathbf{K}) \notin \{2, 3, 5\}$. Let $d: \mathbf{K}^2 \times \mathbf{K}^2 \rightarrow \mathbf{K}$ denote the Lorentz-Minkowski distance defined by $d((x_1, x_2), (y_1, y_2)) = (x_1 - y_1) \cdot (x_2 - y_2)$. H. Schaeffer proved in [7, Satz 1, Satz 2, Satz 3]:

(15) if $f: \mathbf{K}^2 \rightarrow \mathbf{K}^2$ preserves the Lorentz-Minkowski distance 1, $f((0,0)) = (0,0)$ and $f((1,1)) = (1,1)$, then there exists a field homomorphism $\sigma: \mathbf{K} \rightarrow \mathbf{K}$ satisfying $\forall x_1, x_2 \in \mathbf{K} f((x_1, x_2)) = (\sigma(x_1), \sigma(x_2))$ or $\forall x_1, x_2 \in \mathbf{K} f((x_1, x_2)) = (\sigma(x_2), \sigma(x_1))$.

Unfortunately, the proof of Satz 3 in [7] is complicated, the main part of this proof was constructed using computer software.

Let $\varphi_{\mathbf{K}}: \mathbf{K}^2 \times \mathbf{K}^2 \rightarrow \mathbf{K}$, $\varphi_{\mathbf{K}}((x_1, x_2), (y_1, y_2)) = (x_1 - y_1)^2 + (x_2 - y_2)^2$. Theorem 4 generalizes Theorem 3.

Theorem 4. *Let there exists $i \in \mathbf{K}$ such that $i^2 + 1 = 0$. Let $f: \mathbf{K}^2 \rightarrow \mathbf{K}^2$ preserves unit distance defined by $\varphi_{\mathbf{K}}$. We claim that f has a form $I \circ (\sigma, \sigma)$, where $\sigma: \mathbf{K} \rightarrow \mathbf{K}$ is a field homomorphism and $I: \mathbf{K}^2 \rightarrow \mathbf{K}^2$ is an affine mapping with orthogonal linear part.*

Proof. Assume that $f((0,0)) = (0,0)$. The mappings

$$\mathbf{K}^2 \ni (x_1, x_2) \xrightarrow{\xi} (x_1 + i \cdot x_2, x_1 - i \cdot x_2) \in \mathbf{K}^2$$

and

$$\mathbf{K}^2 \ni (x_1, x_2) \xrightarrow{\eta} \left(\frac{1}{2}x_1 + \frac{1}{2}x_2, -\frac{i}{2}x_1 + \frac{i}{2}x_2 \right) \in \mathbf{K}^2$$

satisfy:

$$\eta \circ \xi = \xi \circ \eta = \text{id}(\mathbf{K}^2),$$

$$\forall x_1, x_2, y_1, y_2 \in \mathbf{K} \varphi_{\mathbf{K}}((x_1, x_2), (y_1, y_2)) = d(\xi((x_1, x_2)), \xi((y_1, y_2))),$$

$$\forall x_1, x_2, y_1, y_2 \in \mathbf{K} d((x_1, x_2), (y_1, y_2)) = \varphi_{\mathbf{K}}(\eta((x_1, x_2)), \eta((y_1, y_2))).$$

Therefore, $\xi \circ f \circ \eta: \mathbf{K}^2 \rightarrow \mathbf{K}^2$ preserves the Lorentz-Minkowski distance 1. Obviously, $(\xi \circ f \circ \eta)((0,0)) = (0,0)$. Let $(\xi \circ f \circ \eta)((1,1)) = (a,b) \in \mathbf{K}^2$. We have: $1 = d((1,1), (0,0)) = d((\xi \circ f \circ \eta)((1,1)), (\xi \circ f \circ \eta)((0,0))) = d((a,b), (0,0)) = a \cdot b$. Hence $b = \frac{1}{a}$. For each $z \in \mathbf{K} \setminus \{0\}$ the mapping

$$\mathbf{K}^2 \ni (x, y) \xrightarrow{\lambda(z)} \left(\frac{x}{z}, z \cdot y \right) \in \mathbf{K}^2$$

preserves all Lorentz-Minkowski distances, $\lambda(\frac{1}{z}) \circ \lambda(z) = \lambda(z) \circ \lambda(\frac{1}{z}) = \text{id}(\mathbf{K}^2)$. The mapping $\lambda(a) \circ \xi \circ f \circ \eta: \mathbf{K}^2 \rightarrow \mathbf{K}^2$ preserves the Lorentz-Minkowski distance 1, $(\lambda(a) \circ \xi \circ f \circ \eta)((0,0)) = (0,0)$ and $(\lambda(a) \circ \xi \circ f \circ \eta)((1,1)) = (1,1)$. By (15) there exists a field homomorphism $\sigma: \mathbf{K} \rightarrow \mathbf{K}$ satisfying

$$\underbrace{\lambda(a) \circ \xi \circ f \circ \eta = (\sigma, \sigma)}_{\text{case 1}} \quad \text{or} \quad \underbrace{\lambda(a) \circ \xi \circ f \circ \eta = h \circ (\sigma, \sigma)}_{\text{case 2}},$$

where $h: \mathbf{K}^2 \rightarrow \mathbf{K}^2$, $h((x_1, x_2)) = (x_2, x_1)$.

In case 1: $f = \eta \circ \lambda(\frac{1}{a}) \circ (\sigma, \sigma) \circ \xi = f_1 \circ (\sigma, \sigma)$, where $f_1: \mathbf{K}^2 \rightarrow \mathbf{K}^2$,

$$\begin{aligned} f_1((x_1, x_2)) = & \left(\left(\frac{a}{2} + \frac{1}{2a} \right) \cdot x_1 + \left(\frac{a}{2} - \frac{1}{2a} \right) \sigma(i) \cdot x_2, \right. \\ & \left. - \left(\frac{a}{2} - \frac{1}{2a} \right) i \cdot x_1 - \left(\frac{a}{2} + \frac{1}{2a} \right) i \sigma(i) \cdot x_2 \right). \end{aligned}$$

In case 2: $f = \eta \circ \lambda(\frac{1}{a}) \circ h \circ (\sigma, \sigma) \circ \xi = f_2 \circ (\sigma, \sigma)$, where $f_2: K^2 \rightarrow K^2$,

$$f_2((x_1, x_2)) = \left(\left(\frac{a}{2} + \frac{1}{2a} \right) \cdot x_1 - \left(\frac{a}{2} - \frac{1}{2a} \right) \sigma(i) \cdot x_2, \right. \\ \left. - \left(\frac{a}{2} - \frac{1}{2a} \right) i \cdot x_1 + \left(\frac{a}{2} + \frac{1}{2a} \right) i \sigma(i) \cdot x_2 \right).$$

The mappings f_1 and f_2 are linear and orthogonal. The proof is completed. $\square$

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TECHNICAL FACULTY,
HUGO KOLŁĄTAJ UNIVERSITY,
BALICKA 104, 30-149 KRAKÓW,
POLAND
E-mail address: rttyszka@cyf-kr.edu.pl