

WARPED PRODUCT SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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ABSTRACT. B.Y. Chen [5] established a sharp inequality for the warping function of a warped product submanifold in a Riemannian space form in terms of the squared mean curvature. Later, in [4], he studied warped product submanifolds in complex hyperbolic spaces.

In the present paper, we establish an inequality between the warping function f (intrinsic structure) and the squared mean curvature $\|H\|^2$ and the holomorphic sectional curvature c (extrinsic structures) for warped product submanifolds $M_1 \times_f M_2$ in any generalized complex space form $\tilde{M}(c, \alpha)$.

INTRODUCTION

The notion of *warped product* plays some important role in differential geometry as well as in physics [3]. For instance, the best relativistic model of the Schwarzschild space-time that describes the out space around a massive star or a black hole is given as a warped product.

One of the most fundamental problems in the theory of submanifolds is the immersibility (or non-immersibility) of a Riemannian manifold in a Euclidean space (or, more generally, in a space form). According to a well-known theorem on Nash, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with sufficiently high codimension.

Nash's theorem implies, in particular, that every warped product $M_1 \times_f M_2$ can be immersed as a Riemannian submanifold in some Euclidean space. Moreover, many important submanifolds in real and complex space forms are expressed as a warped product submanifold.

Every Riemannian manifold of constant curvature c can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$. For example, $S^n(1)$ is locally isometric to $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} S^{n-1}(1)$, $\mathbf{E}^n$ is locally isometric to $(0, \infty) \times_x S^{n-1}(1)$ and $H^n(-1)$ is locally isometric to $\mathbf{R} \times_{e^x} \mathbf{E}^{n-1}$ (see [3]).

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1. PRELIMINARIES

Let $\widetilde{M}$ be an almost Hermitian manifold with almost complex structure J and Riemannian metric g . One denotes by $\widetilde{\nabla}$ the operator of covariant differentiation with respect to g on $\widetilde{M}$.

Definition 1.1. If the almost complex structure J satisfies

$$(\widetilde{\nabla}_X J)Y + (\widetilde{\nabla}_Y J)X = 0,$$

for any vector fields X and Y on $\widetilde{M}$, then the manifold $\widetilde{M}$ is called a *nearly-Kaehler manifold* [10].

Remark 1.2. The above condition is equivalent to

$$(\widetilde{\nabla}_X J)X = 0, \quad \forall X \in \Gamma T\widetilde{M}.$$

For an almost complex structure J on the manifold M , the *Nijenhuis tensor field* is defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

for any vector fields X, Y tangent to M , where $[\cdot, \cdot]$ is the Lie bracket.

A necessary and sufficient condition for a nearly-Kaehler manifold to be Kaehler is the vanishing of the Nijenhuis tensor N_J . Any 4-dimensional nearly-Kaehler manifold is a Kaehler manifold.

Example 1.3. Let S^6 be the 6-dimensional unit sphere defined as follows. Let $\mathbf{E}^7$ be the set of all purely imaginary Cayley numbers. Then $\mathbf{E}^7$ is a 7-dimensional subspace of the Cayley algebra C . Let $\{1 = e_0, e_1, \dots, e_6\}$ be a basis of the Cayley algebra, 1 being the unit element of C . If $X = \sum_{i=0}^6 x^i e_i$ and $Y = \sum_{i=0}^6 y^i e_i$ are two elements of $\mathbf{E}^7$, one defines the *scalar product* in $\mathbf{E}^7$ by

$$\langle X, Y \rangle = \sum_{i=0}^6 x^i y^i,$$

and the *vector product* by

$$X \times Y = \sum_{i \neq j} x^i y^j e_i * e_j,$$

$*$ being the multiplication operation of C .

Consider the 6-dimensional unit sphere S^6 in $\mathbf{E}^7$:

$$S^6 = \{X \in \mathbf{E}^7 \mid \langle X, X \rangle = 1\}.$$

The scalar product in $\mathbf{E}^7$ induces the natural metric tensor field g on S^6 . The tangent space $T_X S^6$ at $X \in S^6$ can naturally be identified with the subspace of $\mathbf{E}^7$ orthogonal to X . Define the endomorphism J_X on $T_X S^6$ by

$$J_X Y = X \times Y, \text{ for } Y \in T_X S^6.$$

It is easy to see that

$$g(J_X Y, J_X Z) = g(Y, Z), \quad Y, Z \in T_X S^6.$$

The correspondence $X \mapsto J_X$ defines a tensor field J such that $J^2 = -I$. Consequently, S^6 admits an almost Hermitian structure (J, g) . This structure is a non-Kaehlerian nearly-Kaehlerian structure (its Betti numbers of even order are 0).

We will consider a class of almost Hermitian manifolds, called *RK-manifolds*, which contains nearly-Kaehler manifolds.

Definition 1.4 ([9]). An *RK-manifold* $(\widetilde{M}, J, g)$ is an almost Hermitian manifold for which the curvature tensor $\widetilde{R}$ is invariant by J , i.e.

$$\widetilde{R}(JX, JY, JZ, JW) = \widetilde{R}(X, Y, Z, W),$$

for any $X, Y, Z, W \in \Gamma T\widetilde{M}$.

An almost Hermitian manifold $\widetilde{M}$ is of *pointwise constant type* if for any $p \in \widetilde{M}$ and $X \in T_p\widetilde{M}$ we have $\lambda(X, Y) = \lambda(X, Z)$, where

$$\lambda(X, Y) = \widetilde{R}(X, Y, JX, JY) - \widetilde{R}(X, Y, X, Y)$$

and Y and Z are unit tangent vectors on $\widetilde{M}$ at p , orthogonal to X and JX , i.e. $g(X, X) = g(Y, Y) = 1$, $g(X, Y) = g(JX, Y) = g(X, Z) = g(JX, Z) = 0$.

The manifold $\widetilde{M}$ is said to be of *constant type* if for any unit $X, Y \in \Gamma T\widetilde{M}$ with $g(X, Y) = g(JX, Y) = 0$, $\lambda(X, Y)$ is a constant function.

Recall the following result [9].

Theorem 1.5. *Let $\widetilde{M}$ be an RK-manifold. Then $\widetilde{M}$ is of pointwise constant type if and only if there exists a function α on $\widetilde{M}$ such that*

$$\lambda(X, Y) = \alpha[g(X, X)g(Y, Y) - (g(X, Y))^2 - (g(X, JY))^2],$$

for any $X, Y \in \Gamma T\widetilde{M}$.

Moreover, $\widetilde{M}$ is of constant type if and only if the above equality holds good for a constant α .

In this case, α is the *constant type* of $\widetilde{M}$.

Definition 1.6. A *generalized complex space form* is an RK-manifold of constant holomorphic sectional curvature and of constant type.

We will denote a generalized complex space form by $\widetilde{M}(c, \alpha)$, where c is the constant holomorphic sectional curvature and α the constant type, respectively.

Each complex space form is a generalized complex space form. The converse statement is not true. The sphere S^6 endowed with the standard nearly-Kaehler structure is an example of generalized complex space form which is not a complex space form.

Let $\widetilde{M}(c, \alpha)$ be a generalized complex space form of constant holomorphic sectional curvature c and of constant type α . Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}(c, \alpha)$ has the following expression [9]:

$$(1.1) \quad \begin{aligned} \widetilde{R}(X, Y)Z &= \frac{c+3\alpha}{4}[g(Y, Z)X - g(X, Z)Y] \\ &\quad + \frac{c-\alpha}{4}[g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ]. \end{aligned}$$

Let M be an n -dimensional submanifold of a $2m$ -dimensional generalized complex space form $\widetilde{M}(c, \alpha)$ of constant holomorphic sectional curvature c and constant type α . One denotes by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$, and ∇ the Riemannian connection of M , respectively.

Also, let h be the second fundamental form and R the Riemann curvature tensor of M . Then the equation of Gauss is given by

$$(1.2) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) \\ &+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned}$$

for any vectors X, Y, Z, W tangent to M .

Let $p \in M$ and $\{e_1, \dots, e_n, \dots, e_{2m}\}$ an orthonormal basis of the tangent space $T_p \widetilde{M}(c, \alpha)$, such that $e_1, \dots, e_n$ are tangent to M at p . We denote by H the mean curvature vector, that is

$$(1.3) \quad H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i).$$

Also, we set

$$(1.4) \quad h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m\}.$$

and

$$(1.5) \quad \|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).$$

For any tangent vector field X to M , we put $JX = PX + FX$, where PX and FX are the tangential and normal components of JX , respectively. We denote by

$$(1.6) \quad \|P\|^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j).$$

Let M be a Riemannian n -manifold and $\{e_1, \dots, e_n\}$ be an orthonormal frame field on M . For a differentiable function f on M , the Laplacian Δf of f is defined by

$$(1.7) \quad \Delta f = \sum_{j=1}^n \{(\nabla_{e_j} e_j) f - e_j e_j f\}.$$

We recall the following result of Chen for later use.

Lemma 1.7 ([1]). *Let $n \geq 2$ and $a_1, \dots, a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^n a_i \right)^2 = (n-1) \left(\sum_{i=1}^n a_i^2 + b \right)$$

Then $2a_1 a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = \dots = a_n.$$

2. WARPED PRODUCT SUBMANIFOLDS

Chen established a sharp relationship between the warping function f of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$ (see [5]). In [7], we established a relationship between the warping function f of a warped product $M_1 \times_f M_2$ isometrically immersed in a complex space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1 . The *warped product* of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ (see, for instance, [5]).

Let $x: M_1 \times_f M_2 \rightarrow \widetilde{M}(c, \alpha)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a generalized complex space form $\widetilde{M}(c, \alpha)$. We denote by h the second fundamental form of x and $H_i = \frac{1}{n_i} \text{trace } h_i$, where $\text{trace } h_i$ is the trace of h restricted to M_i and $n_i = \dim M_i$ ($i = 1, 2$).

For a warped product $M_1 \times_f M_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibres, respectively. Thus, $\mathcal{D}_1$ is obtained from the tangent vectors of M_1 via the horizontal lift and $\mathcal{D}_2$ by tangent vectors of M_2 via the vertical lift.

Let $M_1 \times_f M_2$ be a warped product submanifold of a generalized complex space form $\widetilde{M}(c, \alpha)$ of constant holomorphic sectional curvature c and constant type α .

Since $M_1 \times_f M_2$ is a warped product, it is known that

$$(2.1) \quad \nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z,$$

for any vector fields X, Z tangent to M_1, M_2 , respectively.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$(2.2) \quad K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}.$$

We choose a local orthonormal frame

$$\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}\},$$

such that $e_1, \dots, e_{n_1}$ are tangent to M_1 , $e_{n_1+1}, \dots, e_n$ are tangent to M_2 , e_{n+1} is parallel to the mean curvature vector H .

Then, using (2.2), we get

$$(2.3) \quad \frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s),$$

for each $s \in \{n_1 + 1, \dots, n\}$.

From the equation of Gauss, we have

$$(2.4) \quad n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1) \frac{c+3\alpha}{4} - 3\|P\|^2 \frac{c-\alpha}{4}.$$

We set

$$(2.5) \quad \delta = 2\tau - n(n-1) \frac{c+3\alpha}{4} - 3\|P\|^2 \frac{c-\alpha}{4} - \frac{n^2}{2} \|H\|^2.$$

Then, (2.4) can be written as

$$(2.6) \quad n^2 \|H\|^2 = 2(\delta + \|h\|^2).$$

With respect to the above orthonormal frame, (2.6) takes the following form:

$$\left(\sum_{i=1}^n h_{ii}^{n+1} \right)^2 = 2 \left\{ \delta + \sum_{i=1}^n (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2 \right\}.$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^n h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^3 a_i\right)^2 = 2 \left\{ \delta + \sum_{i=1}^3 a_i^2 + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2 \right. \\ \left. - \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} \right\}.$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.

$$\left(\sum_{i=1}^3 a_i\right)^2 = 2 \left(b + \sum_{i=1}^3 a_i^2\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$. In the case under consideration, this means

$$(2.7) \quad \sum_{1 \leq j < k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} + \sum_{n_1+1 \leq s < t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} \\ \geq \frac{\delta}{2} + \sum_{1 \leq \alpha < \beta \leq n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{\alpha, \beta=1}^n (h_{\alpha\beta}^r)^2.$$

Equality holds if and only if

$$(2.8) \quad \sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^n h_{tt}^{n+1}.$$

Using again the Gauss equation, we have

$$(2.9) \quad n_2 \frac{\Delta f}{f} = \tau - \sum_{1 \leq j < k \leq n_1} K(e_j \wedge e_k) - \sum_{n_1+1 \leq s < t \leq n} K(e_s \wedge e_t) = \\ = \tau - \frac{n_1(n_1-1)(c+3\alpha)}{8} - \sum_{r=n+1}^{2m} \sum_{1 \leq j < k \leq n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) \\ - 3 \frac{c-\alpha}{4} \sum_{1 \leq j < k \leq n_1} g^2(Je_j, e_k) - \frac{n_2(n_2-1)(c+3\alpha)}{8} \\ - \sum_{r=n+1}^{2m} \sum_{n_1+1 \leq s < t \leq n} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2) - 3 \frac{c-\alpha}{4} \sum_{n_1+1 \leq s < t \leq n} g^2(Je_s, e_t).$$

Combining (2.7) and (2.9) and taking account of (2.3), we obtain

$$(2.10) \quad n_2 \frac{\Delta f}{f} \leq \tau - \frac{n(n-1)(c+3\alpha)}{8} + n_1 n_2 \frac{c+3\alpha}{4} - \frac{\delta}{2} \\ - 3 \frac{c-\alpha}{4} \sum_{1 \leq j < k \leq n_1} g^2(Je_j, e_k) - 3 \frac{c-\alpha}{4} \sum_{n_1+1 \leq s < t \leq n} g^2(Je_s, e_t) \\ - \sum_{1 \leq j \leq n_1; n_1+1 \leq t \leq n} (h_{jt}^{n+1})^2 - \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{\alpha, \beta=1}^n (h_{\alpha\beta}^r)^2$$

$$\begin{aligned}
 & + \sum_{r=n+2}^{2m} \sum_{1 \leq j < k \leq n_1} ((h_{jk}^r)^2 - h_{jj}^r h_{kk}^r) + \sum_{r=n+2}^{2m} \sum_{n_1+1 \leq s < t \leq n} ((h_{st}^r)^2 - h_{ss}^r h_{tt}^r) \\
 & = \tau - \frac{n(n-1)(c+3\alpha)}{8} + n_1 n_2 \frac{c+3\alpha}{4} - \frac{\delta}{2} - \sum_{r=n+1}^{2m} \sum_{1 \leq j \leq n_1; n_1+1 \leq t \leq n} (h_{jt}^r)^2 \\
 & \quad - 3 \frac{c-\alpha}{4} \sum_{1 \leq j < k \leq n_1} g^2(Je_j, e_k) - 3 \frac{c-\alpha}{4} \sum_{n_1+1 \leq s < t \leq n} g^2(Je_s, e_t) \\
 & \quad - \frac{1}{2} \sum_{r=n+2}^{2m} \left(\sum_{j=1}^{n_1} h_{jj}^r \right)^2 - \frac{1}{2} \sum_{r=n+2}^{2m} \left(\sum_{t=n_1+1}^n h_{tt}^r \right)^2 \\
 & \leq \tau - \frac{n(n-1)(c+3\alpha)}{8} + n_1 n_2 \frac{c+3\alpha}{4} - \frac{\delta}{2} - 3 \frac{c-\alpha}{4} \sum_{1 \leq j < k \leq n_1} g^2(Je_j, e_k) \\
 & \quad - 3 \frac{c-\alpha}{4} \sum_{n_1+1 \leq s < t \leq n} g^2(Je_s, e_t).
 \end{aligned}$$

The equality sign of (2.10) holds if and only if

$$(2.10.1) \quad h_{jt}^r = 0, \quad 1 \leq j \leq n_1, n_1+1 \leq t \leq n, n+1 \leq r \leq 2m,$$

and

$$(2.10.2) \quad \sum_{i=1}^{n_1} h_{ii}^r = \sum_{t=n_1+1}^n h_{tt}^r = 0, \quad n+2 \leq r \leq 2m.$$

Obviously (2.10.1) is equivalent to the mixed totally geodesicness of the warped product $M_1 \times_f M_2$ (i.e. $h(X, Z) = 0$, for any X in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$) and (2.8) and (2.10.2) imply $n_1 H_1 = n_2 H_2$.

Using (2.5), we finally obtain

Lemma 2.1. *Let $x: M_1 \times_f M_2 \rightarrow \widetilde{M}(c, \alpha)$ be an isometric immersion of an n -dimensional warped product into a $2m$ -dimensional generalized complex space form $\widetilde{M}(c, \alpha)$. Then:*

$$(2.11) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \frac{c+3\alpha}{4} + 3 \frac{c-\alpha}{4n_2} \sum_{1 \leq i \leq n_1} \sum_{n_1+1 \leq s \leq n} g^2(Je_i, e_s).$$

where $n_i = \dim M_i, i = 1, 2$, and Δ is the Laplacian operator of M_1 .

From the above Lemma, it follows

Theorem 2.2. *Let $x: M_1 \times_f M_2 \rightarrow \widetilde{M}(c, \alpha)$ be an isometric immersion of an n -dimensional warped product into a $2m$ -dimensional generalized complex space form $\widetilde{M}(c, \alpha)$. Then:*

i) *If $c < \alpha$, then*

$$(2.12) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \frac{c+3\alpha}{4}.$$

Moreover, the equality case of (2.12) holds identically if and only if x is a mixed totally geodesic immersion, $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors and $J\mathcal{D}_1 \perp \mathcal{D}_2$.

ii) If $c = \alpha$, then

$$(2.13) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \frac{c + 3\alpha}{4}.$$

Moreover, the equality case of (2.13) holds identically if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i ($i = 1, 2$), are the partial mean curvature vectors.

iii) If $c > \alpha$, then

$$(2.14) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \frac{c + 3\alpha}{4} + 3 \frac{c - \alpha}{8} \|P\|^2.$$

Moreover, the equality case of (2.14) holds identically if and only if x is a mixed totally geodesic immersion, $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors and both M_1 and M_2 are totally real submanifolds.

A submanifold N in a Kaehler manifold $\widetilde{M}$ is called a *CR-submanifold* if there exists on N a holomorphic distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^\perp$ is a totally real distribution, i.e., $J\mathcal{D}_x^\perp \subset T_x^\perp N$.

A CR-submanifold of a Kaehler manifold $\widetilde{M}$ is called a *CR-product* if it is a Riemannian product of a Kaehler submanifold and a totally real submanifold.

There do not exist warped product CR-submanifolds of the form $M_\perp \times_f M_\top$, with $M_\perp$ a totally real submanifold and $M_\top$ a complex submanifold, other than CR-products. A *CR-warped product* is a warped product CR-submanifold of the form $M_\top \times_f M_\perp$, by reversing the two factors [2].

Obviously, any CR-warped product submanifold, in particular any CR-product, satisfies $J\mathcal{D}_1 \perp \mathcal{D}_2$.

Corollary 2.3. *Let M be an n -dimensional CR-warped product submanifold of a $2m$ -dimensional generalized complex space form $\widetilde{M}(c, \alpha)$. Then:*

$$(2.15) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \frac{c + 3\alpha}{4}.$$

Moreover, the equality case of (2.15) holds identically if and only if x is a mixed totally geodesic immersion, $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors.

We derive the following non-existence results.

Corollary 2.4. *Let $\widetilde{M}(c, \alpha)$ be a generalized complex space form, M_1 an n_1 -dimensional Riemannian manifold and f a differentiable function on M_1 . If there is a point $p \in M_1$ such that $(\Delta f)(p) > n_1 \frac{c+3\alpha}{4} f(p)$, then there do not exist any minimal CR-warped product submanifold $M_1 \times_f M_2$ in $\widetilde{M}(c, \alpha)$.*

Corollary 2.5. *Let $\widetilde{M}(c, \alpha)$ be a generalized complex space form, with $c > \alpha$, M_1 an n_1 -dimensional totally real submanifold of $\widetilde{M}(c, \alpha)$ and f a differentiable function on M_1 . If there is a point $p \in M_1$ such that $(\Delta f)(p) > n_1 \frac{c+3\alpha}{4} f(p)$, then there do not exist any totally real submanifold M_2 in $\widetilde{M}(c, \alpha)$ such that $M_1 \times_f M_2$ be a minimal warped product submanifold in $\widetilde{M}(c, \alpha)$.*

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