

PARTIAL SUMS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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ABSTRACT. In this paper, we study the ratio of a function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

to its sequence of partial sums of the form $f_n(z) = z + \sum_{k=2}^n a_k z^k$. Also, we will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\{f'_n(z)/f'(z)\}$.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form:

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Then a function $f(z)$ belonging to $\mathcal{A}$ is said to be starlike of order α if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathcal{U})$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $\mathcal{S}_\alpha^*$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Also, a function $f(z)$ belonging to $\mathcal{A}$ is said to be convex of order α if it satisfies

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathcal{U})$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $\mathcal{K}_\alpha$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{P}_\alpha$ iff

$$(1.4) \quad \operatorname{Re}(f'(z)) > \alpha, \quad (z \in \mathcal{U}).$$

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It is well known that $\mathcal{K}_\alpha \subset \mathcal{S}_\alpha^* \subset \mathcal{S}$. Given two functions $f, g \in \mathcal{A}$, where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, their *Hadamard product* or *convolution* $f(z) * g(z)$ is defined by

$$(1.5) \quad f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathcal{U}).$$

Ruschewehy [2] using the convolution techniques, introduced and studied the class of prestarlike functions of order α , which is denoted by $\mathcal{R}_\alpha$. Thus $f \in \mathcal{A}$ is said to be prestarlike functions of order α ($0 \leq \alpha < 1$) if $f * s_\alpha(z) \in \mathcal{S}_\alpha^*$ where $s_\alpha(z) = z/(1-z)^{2(1-\alpha)}$. It may be noted that $\mathcal{R}_0 \equiv \mathcal{K}_0$ and $\mathcal{R}_{1/2} \equiv \mathcal{S}_{1/2}^*$.

In our present paper we shall make use of the following definition due to Juneja et al. [2].

Definition. Given the analytic functions

$$(1.6) \quad \Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k \text{ and } \Psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k, \quad (0 \leq \alpha < 1; z \in \mathcal{U}),$$

where $\lambda_k \geq 0, \mu_k \geq 0$ and $\lambda_k \geq \mu_k$ for $k \geq 2$, we say that $f \in \mathcal{A}$ is in $\mathcal{E}(\Phi, \Psi; \alpha)$ if $f(z) * \Psi(z) \neq 0$ and

$$(1.7) \quad \operatorname{Re} \left\{ \frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} \right\} > \alpha \quad (z \in \mathcal{U}).$$

It is easy to check that various subclasses of $\mathcal{S}$ referred to above can be represented as $\mathcal{E}(\Phi, \Psi; \alpha)$ for suitable choices of Φ, Ψ . For example

- (i) $\mathcal{E} \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha \right) = \mathcal{S}_\alpha^*$;
- (ii) $\mathcal{E} \left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha \right) = \mathcal{K}_\alpha$;
- (iii) $\mathcal{E} \left(\frac{z}{(1-z)^2}, z; \alpha \right) = \mathcal{P}_\alpha$;
- (iv) $\mathcal{E} \left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; \alpha \right) = \mathcal{R}_\alpha$.

In fact many new subclasses of $\mathcal{S}$ can be defined and studied by suitably choosing $\Phi(z)$ and $\Psi(z)$. Thus

- (v) $\mathcal{E} \left(\frac{z+z^2}{(1-z)^3}, z; \alpha \right) = \{f \in \mathcal{S}: \operatorname{Re}((zf'(z))') > \alpha\}$
- (vi) $\mathcal{E} \left((1-\delta)\frac{z}{(1-z)^2} + \delta\frac{z+z^2}{(1-z)^3}, z; \alpha \right) = \{f \in \mathcal{S}: \operatorname{Re}((1-\delta)f'(z) + \delta(zf'(z))') > \alpha\}$ and so on.

A sufficient condition for a function of the form (1.1) to be in $\mathcal{E}(\Phi, \Psi; \alpha)$ is that

$$(1.8) \quad \sum_{k=2}^{\infty} (\lambda_k - \alpha \mu_k) |a_k| \leq 1 - \alpha.$$

For the functions of the form

$$(1.9) \quad f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0$$

the sufficient condition (1.8) is also necessary (see [1]).

In the present paper and by following the earlier works by Silverman [3] on partial sums of analytic functions, we study the ratio of a function of the form (1.1) to its sequence of partial sums of the form

$$(1.10) \quad f_n(z) = z + \sum_{k=2}^n a_k z^k.$$

when the coefficients of $f(z)$ are satisfy the condition (1.8). We will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\{f'_n(z)/f'(z)\}$. It is seen that this study not only gives as a particular case, the results of Silverman [3] but also give rise to several new results.

2. MAIN RESULTS

Theorem 1. *If $f(z)$ of the form (1.1) satisfies the condition (1.8), and*

$$\lambda_{k+1} - \alpha\mu_{k+1} \geq \begin{cases} 1 - \alpha, & k = 2, 3, \dots, n \\ \lambda_{n+1} + \alpha\mu_{n+1}, & k = n+1, n+2, \dots \end{cases}$$

then

$$(2.1) \quad \operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} \quad (z \in \mathcal{U})$$

and

$$(2.2) \quad \operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha + \lambda_{n+1} - \alpha\mu_{n+1}} \quad (z \in \mathcal{U}).$$

The results (2.1) and (2.2) are sharp with the function given by

$$(2.3) \quad f(z) = z + \frac{1 - \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} z^{n+1}.$$

Proof. Define the function $w(z)$ by

$$(2.4) \quad \begin{aligned} \frac{1+w(z)}{1-w(z)} &= \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \left[\frac{f(z)}{f_n(z)} - \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} \right) \right] \\ &= \frac{1 + \sum_{k=2}^n a_k z^{k-1} + \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^n a_k z^{k-1}}. \end{aligned}$$

It suffices to show that $|w(z)| \leq 1$. Now, from (2.4) we can write

$$w(z) = \frac{\left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} a_k z^{k-1}}{2 + 2 \sum_{k=2}^n a_k z^{k-1} + \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} a_k z^{k-1}}$$

to find that

$$|w(z)| \leq \frac{\left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^n |a_k| - \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} |a_k|}$$

Now $|w(z)| \leq 1$ if

$$2 \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} |a_k| \leq 2 - 2 \sum_{k=2}^n |a_k|$$

or, equivalently

$$\sum_{k=2}^n |a_k| + \sum_{k=n+1}^{\infty} \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} |a_k| \leq 1.$$

From the condition (1.8), it is sufficient to show that

$$\sum_{k=2}^n |a_k| + \sum_{k=n+1}^{\infty} \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} |a_k| \leq \sum_{k=2}^{\infty} \frac{\lambda_k - \alpha\mu_k}{1 - \alpha} |a_k|$$

which is equivalent to

$$(2.5) \quad \sum_{k=2}^n \left(\frac{\lambda_k - \alpha\mu_k - 1 + \alpha}{1 - \alpha} \right) |a_k| + \sum_{k=n+1}^{\infty} \left(\frac{\lambda_k - \alpha\mu_k - \lambda_{n+1} + \alpha\mu_{n+1}}{1 - \alpha} \right) |a_k| \geq 0.$$

To see that the function given by (2.3) gives the sharp result, we observe that

$$\begin{aligned} \frac{f(z)}{f_n(z)} &= 1 + \frac{1 - \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} z^n \rightarrow 1 - \frac{1 - \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} \\ &= \frac{\lambda_{n+1} - \alpha\mu_{n+1} - 1 + \alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} \quad \text{when } r \rightarrow 1^-. \end{aligned}$$

To prove the second part of this theorem, we write

$$\begin{aligned} \frac{1 + w(z)}{1 - w(z)} &= \frac{1 - \alpha + \lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \left[\frac{f_n(z)}{f(z)} - \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha + \lambda_{n+1} - \alpha\mu_{n+1}} \right) \right] \\ &= \frac{1 + \sum_{k=2}^n a_k z^{k-1} - \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^n a_k z^{k-1}} \end{aligned}$$

where

$$|w(z)| \leq \frac{\left(\frac{1 - \alpha + \lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^n |a_k| - \left(\frac{1 - \alpha - \lambda_{n+1} + \alpha\mu_{n+1}}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} |a_k|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^n |a_k| + \sum_{k=n+1}^{\infty} \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{1 - \alpha} |a_k| \leq 1.$$

Making use of (1.8) to get (2.5). Finally, equality holds in (2.2) for the extremal function $f(z)$ given by (2.3). $\square$

Taking $\Phi(z) = z/(1 - z)^2$ and $\Psi(z) = z/(1 - z)$ in Theorem 1, we obtain

Corollary 1 ([3]). *Let the function $f(z)$ be defined by (1.1). If*

$$(2.6) \quad \sum_{k=2}^{\infty} (k - \alpha) |a_k| \leq 1 - \alpha$$

then

$$(2.7) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{n}{n+1-\alpha} \quad (z \in \mathcal{U})$$

and

$$(2.8) \quad \operatorname{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{n+1-\alpha}{n+2-2\alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.9) \quad f(z) = z + \frac{1-\alpha}{n+1-\alpha} z^{n+1}.$$

Taking $\Phi(z) = (z+z^2)/(1-z)^3$ and $\Psi(z) = z/(1-z)^2$ in Theorem 1, we obtain

Corollary 2 ([5]). *Let the function $f(z)$ be defined by (1.1). If*

$$(2.10) \quad \sum_{k=2}^{\infty} k(k - \alpha) |a_k| \leq 1 - \alpha$$

then

$$(2.11) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{n(n+2-\alpha)}{(n+1)(n+1-\alpha)} \quad (z \in \mathcal{U})$$

and

$$(2.12) \quad \operatorname{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{(n+1)(n+1-\alpha)}{(n+1)[(n+1)-\alpha] + 1 - \alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.13) \quad f(z) = z + \frac{1-\alpha}{(n+1)^2 - \alpha(n+1)} z^{n+1}.$$

Taking $\Phi(z) = z/(1-z)$ and $\Psi(z) = z$ in Theorem 1, we obtain

Corollary 3. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.14) \quad \sum_{k=2}^{\infty} |a_k| \leq 1 - \alpha$$

then

$$(2.15) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \alpha \quad (z \in \mathcal{U})$$

and

$$(2.16) \quad \operatorname{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{1}{2-\alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.17) \quad f(z) = z + (1-\alpha)z^{n+1}.$$

Taking $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z$ in Theorem 1, we obtain

Corollary 4. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.18) \quad \sum_{k=2}^{\infty} k |a_k| \leq 1 - \alpha$$

then

$$(2.19) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{n + \alpha}{n + 1} \quad (z \in \mathcal{U})$$

and

$$(2.20) \quad \operatorname{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{n + 1}{n + 2 - \alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.21) \quad f(z) = z + \frac{1 - \alpha}{n + 1} z^{n+1}.$$

Taking $\Phi(z) = (z + (1 - 2\alpha)z^2)/(1 - z)^{3-2\alpha}$ and $\Psi(z) = z/(1 - z)^{2-2\alpha}$ in Theorem 1, we obtain

Corollary 5. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.22) \quad \sum_{k=2}^{\infty} C(\alpha, k)(k - \alpha) |a_k| \leq 1 - \alpha$$

then

$$(2.23) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{C(\alpha, n + 1)(n + 1 - \alpha) - 1 + \alpha}{C(\alpha, n + 1)(n + 1 - \alpha)} \quad (z \in \mathcal{U})$$

and

$$(2.24) \quad \operatorname{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{C(\alpha, n + 1)(n + 1 - \alpha)}{1 - \alpha + C(\alpha, n + 1)(n + 1 - \alpha)} \quad (z \in \mathcal{U}).$$

where $C(\alpha, k) = \prod_{i=2}^k (i - 2\alpha)/(i - 1)!$.

The results are sharp with the function given by

$$(2.25) \quad f(z) = z + \frac{1 - \alpha}{C(\alpha, n + 1)(n + 1 - \alpha)} z^{n+1}.$$

Taking $\Phi(z) = (z + z^2)/(1 - z)^3$ and $\Psi(z) = z$ in Theorem 1, we obtain

Corollary 6. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.26) \quad \sum_{k=2}^{\infty} k^2 |a_k| \leq 1 - \alpha$$

then

$$(2.27) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{n^2 + 2n + \alpha}{(n + 1)^2} \quad (z \in \mathcal{U})$$

and

$$(2.28) \quad \operatorname{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{(n + 1)^2}{n^2 + 2n + 2 - \alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.29) \quad f(z) = z + \frac{1-\alpha}{(n+1)^2} z^{n+1}.$$

Taking $\Phi(z) = (1-\delta)z/(1-z)^2 + \delta(z+z^2)/(1-z)^3$ and $\Psi(z) = z$ in Theorem 1, we obtain

Corollary 7. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.30) \quad \sum_{k=2}^{\infty} [(1-\delta)k + \delta k^2] |a_k| \leq 1 - \alpha$$

then

$$(2.31) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{(n+1)(1+n\delta) - 1 + \alpha}{(n+1)(1+n\delta)} \quad (z \in \mathcal{U})$$

and

$$(2.32) \quad \operatorname{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{(n+1)(1+n\delta)}{1-\alpha + (n+1)(1+n\delta)} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.33) \quad f(z) = z + \frac{1-\alpha}{(n+1)(1+n\delta)} z^{n+1}.$$

We next turn to ratios involving derivatives.

Theorem 2. *If $f(z)$ of the form (1.1) satisfies the condition (1.8), and*

$$\lambda_{k+1} - \alpha\mu_{k+1} \geq \begin{cases} k(1-\alpha), & k = 2, 3, \dots, n \\ k(1-\alpha) + \frac{(\lambda_{n+1}-\alpha\mu_{n+1})k}{(n+1)}, & k = n+1, n+2, \dots \end{cases}$$

then

$$(2.34) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1} - (n+1)(1-\alpha)}{\lambda_{n+1} - \alpha\mu_{n+1}} \quad (z \in \mathcal{U})$$

and

$$(2.35) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha) + \lambda_{n+1} - \alpha\mu_{n+1}} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by (2.3).

Proof. We write

$$\frac{1+w(z)}{1-w(z)} = \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)} \left[\frac{f'(z)}{f'_n(z)} - \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1} - (n+1)(1-\alpha)}{\lambda_{n+1} - \alpha\mu_{n+1}} \right) \right]$$

where

$$w(z) = \frac{\left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)} \right) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{2 + 2 \sum_{k=2}^n k a_k z^{k-1} + \left(\frac{\lambda_{n+1}-\alpha\mu_{n+1}}{(n+1)(1-\alpha)} \right) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}$$

Now $|w(z)| \leq 1$ if

$$\sum_{k=2}^n k |a_k| + \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)} \sum_{k=n+1}^{\infty} k |a_k| \leq 1.$$

From the condition (1.8), it is sufficient to show that

$$\sum_{k=2}^n k |a_k| + \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)} \sum_{k=n+1}^{\infty} k |a_k| \leq \sum_{k=2}^{\infty} \frac{\lambda_k - \alpha\mu_k}{1-\alpha} |a_k|$$

which is equivalent to

$$\begin{aligned} \sum_{k=2}^n \left(\frac{\lambda_k - \alpha\mu_k - (1-\alpha)k}{1-\alpha} \right) |a_k| \\ + \sum_{k=n+1}^{\infty} \frac{(n+1)(\lambda_k - \alpha\mu_k) - (\lambda_{n+1} - \alpha\mu_{n+1})k}{(1-\alpha)(n+1)} |a_k| \geq 0. \end{aligned}$$

To prove the result (2.32), define the function $w(z)$ by

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} = \frac{(n+1)(1-\alpha) + \lambda_{n+1} - \alpha\mu_{n+1}}{1-\alpha} \\ \times \left[\frac{f'_n(z)}{f'(z)} - \left(\frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha) + \lambda_{n+1} - \alpha\mu_{n+1}} \right) \right] \end{aligned}$$

where

$$w(z) = \frac{\left(1 + \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)} \right) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{2 + 2 \sum_{k=2}^n k a_k z^{k-1} + \left(1 + \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)} \right) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}.$$

Now $|w(z)| \leq 1$ if

$$(2.36) \quad \sum_{k=2}^n k |a_k| + \left(1 + \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)} \right) \sum_{k=n+1}^{\infty} k |a_k| \leq 1.$$

It suffices to show that the left hand side of (2.36) is bounded above by the condition $\sum_{k=2}^{\infty} ((\lambda_k - \alpha\mu_k)/(1-\alpha)) |a_k|$, which is equivalent to

$$\sum_{k=2}^n \left(\frac{\lambda_k - \alpha\mu_k}{1-\alpha} - k \right) |a_k| + \sum_{k=n+1}^{\infty} \left(\frac{\lambda_k - \alpha\mu_k}{1-\alpha} - \left(1 + \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)} \right) k \right) |a_k| \geq 0.$$

□

Taking $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z/(1-z)$ in Theorem 2, we obtain

Corollary 8 ([3]). *Let the function $f(z)$ be defined by (1.1). If*

$$(2.37) \quad \sum_{k=2}^{\infty} (k-\alpha) |a_k| \leq 1-\alpha$$

then

$$(2.38) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'_n(z)} \right\} \geq \frac{n\alpha}{n+1-\alpha} \quad (z \in \mathcal{U})$$

and

$$(2.39) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{n+1-\alpha}{(n+1)(2-\alpha)-\alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.40) \quad f(z) = z + \frac{1-\alpha}{n+1-\alpha} z^{n+1}.$$

Taking $\Phi(z) = (z+z^2)/(1-z)^3$ and $\Psi(z) = z/(1-z)^2$ in Theorem 2, we obtain

Corollary 9 ([3]). *Let the function $f(z)$ be defined by (1.1). If*

$$(2.41) \quad \sum_{k=2}^{\infty} k(k-\alpha) |a_k| \leq 1-\alpha$$

then

$$(2.42) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq \frac{n}{n+1-\alpha} \quad (z \in \mathcal{U})$$

and

$$(2.43) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{n+1-\alpha}{n+2-2\alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.44) \quad f(z) = z + \frac{1-\alpha}{(n+1)^2 - \alpha(n+1)} z^{n+1}.$$

Taking $\Phi(z) = z/(1-z)$ and $\Psi(z) = z$ in Theorem 2, we obtain

Corollary 10. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.45) \quad \sum_{k=2}^{\infty} |a_k| \leq 1-\alpha$$

then

$$(2.46) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq 1 - (n+1)(1-\alpha) \quad (z \in \mathcal{U})$$

and

$$(2.47) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{1}{(n+1)(1-\alpha) + 1} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.48) \quad f(z) = z + (1-\alpha)z^{n+1}.$$

Taking $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z$ in Theorem 2, we obtain

Corollary 11. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.49) \quad \sum_{k=2}^{\infty} k |a_k| \leq 1-\alpha$$

then

$$(2.50) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq \frac{(n+1)\alpha}{n+1} \quad (z \in \mathcal{U})$$

and

$$(2.51) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{n+1}{(n+1)(2-\alpha)} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.52) \quad f(z) = z + \frac{1-\alpha}{n+1} z^{n+1}$$

Taking $\Phi(z) = (z + (1-2\alpha)z^2)/(1-z)^{3-2\alpha}$ and $\Psi(z) = z/(1-z)^{2-2\alpha}$ in Theorem 2, we obtain

Corollary 12. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.53) \quad \sum_{k=2}^{\infty} C(\alpha, k)(k-\alpha) |a_k| \leq 1-\alpha$$

then

$$(2.54) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq \frac{C(\alpha, n+1)(n+1-\alpha) - (n+1)(1-\alpha)}{C(\alpha, n+1)(n+1-\alpha)} \quad (z \in \mathcal{U})$$

and

$$(2.55) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{C(\alpha, n+1)(n+1-\alpha)}{(n+1)(1-\alpha) + C(\alpha, n+1)(n+1-\alpha)} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.56) \quad f(z) = z + \frac{1-\alpha}{C(\alpha, n+1)(n+1-\alpha)} z^{n+1}.$$

Taking $\Phi(z) = (z + z^2)/(1-z)^3$ and $\Psi(z) = z$ in Theorem 2, we obtain

Corollary 13. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.57) \quad \sum_{k=2}^{\infty} k^2 |a_k| \leq 1-\alpha$$

then

$$(2.58) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq \frac{n+\alpha}{n+1} \quad (z \in \mathcal{U})$$

and

$$(2.59) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{n+1}{n+2-\alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.60) \quad f(z) = z + \frac{1-\alpha}{(n+1)^2} z^{n+1}.$$

Taking $\Phi(z) = (1-\delta)z/(1-z)^2 + \delta(z + z^2)/(1-z)^3$ and $\Psi(z) = z$ in Theorem 2, we obtain

Corollary 14. *Let the function $f(z)$ be defined by (1.1). If*

$$(2.61) \quad \sum_{k=2}^{\infty} [(1-\delta)k + \delta k^2] |a_k| \leq 1-\alpha$$

then

$$(2.62) \quad \operatorname{Re} \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq \frac{n\delta + \alpha}{1+n\delta} \quad (z \in \mathcal{U})$$

and

$$(2.63) \quad \operatorname{Re} \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{1+n\delta}{2+n\delta-\alpha} \quad (z \in \mathcal{U}).$$

The results are sharp with the function given by

$$(2.64) \quad f(z) = z + \frac{1-\alpha}{(n+1)(1+n\delta)} z^{n+1}.$$

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