

ON CERTAIN GENERALIZED CLASS OF NON-BAZILEVIČ FUNCTIONS

ZHIGANG WANG, CHUNYI GAO AND MAOXIN LIAO

ABSTRACT. In this paper, a subclass $N(\lambda, \alpha, A, B)$ of analytic functions is introduced, which is a generalized class of non-Bazilevič functions. The subordination relations, inclusion relations, distortion theorems and inequality properties are discussed by applying differential subordination method.

1. INTRODUCTION

Let H denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Assume that $0 < \alpha < 1$, a function $f(z) \in N(\alpha)$ if and only if $f(z) \in H$ and

$$(1.2) \quad \operatorname{Re} \left\{ f'(z) \left(\frac{z}{f(z)} \right)^{1+\alpha} \right\} > 0, \quad z \in U.$$

$N(\alpha)$ was introduced by Obradović [5] recently, he called this class of functions to be of non-Bazilevič type. Until now, this class was studied in a direction of finding necessary conditions over α that embeds this class into the class of univalent functions or its subclasses, which is still an open problem.

Let $f(z)$ and $F(z)$ be analytic in U . Then we say that the function $f(z)$ is subordinate to $F(z)$ in U , if there exists an analytic function $\omega(z)$ in U such that $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$, denoted $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in U , then the subordination is equivalent to $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [6]).

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Assume that $0 < \alpha < 1, \lambda \in C, -1 \leq B \leq 1, A \neq B, A \in R$, we define the following subclass of H :

$$(1.3) \quad N(\lambda, \alpha, A, B) = \left\{ f(z) \in H : (1 + \lambda) \left(\frac{z}{f(z)} \right)^\alpha - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha \prec \frac{1 + Az}{1 + Bz}, \quad z \in U \right\},$$

where all the powers are principal values, below we apply this agreement. Apparently, $f(z) \in N(\lambda, \alpha, \beta)$ if and only if $f(z) \in H$ and

$$(1.4) \quad \operatorname{Re} \left\{ (1 + \lambda) \left(\frac{z}{f(z)} \right)^\alpha - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha \right\} > \beta, 0 \leq \beta < 1, \quad z \in U.$$

If $\lambda = -1, A = 1, B = -1$, then the class $N(\lambda, \alpha, A, B)$ reduces to the class of non-Bazilevič functions. If $\lambda = -1, A = 1 - 2\beta, B = -1$, then the class $N(\lambda, \alpha, A, B)$ reduces to the class of non-Bazilevič functions of order β ($0 \leq \beta < 1$). The Fekete-Szegő problem of the class $N(-1, \alpha, 1 - 2\beta, -1)$ were considered by N. Tuneski and M. Darus [8]. In this paper, we will discuss the subordination relations, inclusion relations, distortion theorems and inequality properties of $N(\lambda, \alpha, A, B)$.

2. SOME LEMMAS

Lemma 1 ([4]). Let $F(z) = 1 + b_1z + b_2z^2 + \dots$ be analytic in U , $h(z)$ be analytic and convex in U , $h(0) = 1$. If

$$(2.1) \quad F(z) + \frac{1}{c}zF'(z) \prec h(z),$$

where $c \neq 0$ and $\operatorname{Re} c \geq 0$, then

$$F(z) \prec cz^{-c} \int_0^z t^{c-1} h(t) dt \prec h(z),$$

and $cz^{-c} \int_0^z t^{c-1} h(t) dt$ is the best dominant for differential subordination (2.1).

Lemma 2 ([1]). Let $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, then

$$\frac{1 + A_2z}{1 + B_2z} \prec \frac{1 + A_1z}{1 + B_1z}.$$

Lemma 3 ([3]). Let $F(z)$ be analytic and convex in U , $f(z) \in H, g(z) \in H$,

$$f(z) \prec F(z), g(z) \prec F(z), 0 \leq \lambda \leq 1,$$

then

$$\lambda f(z) + (1 - \lambda)g(z) \prec F(z).$$

Lemma 4 ([7]). Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ be analytic in U , $g(z) = \sum_{k=1}^{\infty} b_k z^k$ be analytic and convex in U . If $f(z) \prec g(z)$, then $|a_k| \leq |b_1|$, for $k = 1, 2, \dots$

3. MAIN RESULTS AND THEIR PROOFS

Theorem 1. Let $0 < \alpha < 1, \operatorname{Re} \lambda \geq 0, -1 \leq B \leq 1, A \neq B, A \in R, f(z) \in N(\lambda, \alpha, A, B)$, then

$$\left(\frac{z}{f(z)} \right)^\alpha \prec \frac{\alpha}{\lambda} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{\alpha}{\lambda}-1} du \prec \frac{1 + Az}{1 + Bz}.$$

Proof. Let

$$(3.1) \quad F(z) = \left(\frac{z}{f(z)} \right)^\alpha,$$

then $F(z) = 1 + b_1 z + b_2 z^2 + \dots$ is analytic in U . By taking the derivatives in the both sides of equation (3.1), we have

$$(1 + \lambda) \left(\frac{z}{f(z)} \right)^\alpha - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha = F(z) + \frac{\lambda}{\alpha} z F'(z).$$

Since $f(z) \in N(\lambda, \alpha, A, B)$, we have

$$F(z) + \frac{\lambda}{\alpha} z F'(z) \prec \frac{1 + Az}{1 + Bz}.$$

It is obvious that $h(z) = (1 + Az)/(1 + Bz)$ is analytic and convex in U , $h(0) = 1$. Since $\alpha/\lambda \neq 0$, $\operatorname{Re}\{\alpha/\lambda\} \geq 0$, therefore it follows from Lemma 1 that

$$\left(\frac{z}{f(z)} \right)^\alpha = F(z) \prec \frac{\alpha}{\lambda} z^{-\frac{\alpha}{\lambda}} \int_0^z \frac{1 + At}{1 + Bt} t^{\frac{\alpha}{\lambda}-1} dt = \frac{\alpha}{\lambda} \int_0^1 \frac{1 + Azu}{1 + Bzu} u^{\frac{\alpha}{\lambda}-1} du \prec \frac{1 + Az}{1 + Bz}.$$

□

Corollary 1. Let $0 < \alpha < 1$, $\operatorname{Re} \lambda \geq 0$, $\beta \neq 1$. If

$$(1 + \lambda) \left(\frac{z}{f(z)} \right)^\alpha - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha \prec \frac{1 + (1 - 2\beta)z}{1 - z}, \quad z \in U,$$

then

$$\left(\frac{z}{f(z)} \right)^\alpha \prec \beta + \frac{(1 - \beta)\alpha}{\lambda} \int_0^1 \frac{1 + zu}{1 - zu} u^{\frac{\alpha}{\lambda}-1} du, \quad z \in U.$$

Corollary 2. Let $0 < \alpha < 1$, $\operatorname{Re} \lambda \geq 0$, then

$$N(\lambda, \alpha, A, B) \subset N(0, \alpha, A, B).$$

Theorem 2. Let $0 < \alpha < 1$, $\lambda_2 \geq \lambda_1 \geq 0$, $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, then

$$N(\lambda_2, \alpha, A_2, B_2) \subset N(\lambda_1, \alpha, A_1, B_1).$$

Proof. Let $f(z) \in N(\lambda_2, \alpha, A_2, B_2)$, we have $f(z) \in H$ and

$$(1 + \lambda_2) \left(\frac{z}{f(z)} \right)^\alpha - \lambda_2 \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, therefore it follows from Lemma 2 that

$$(3.2) \quad (1 + \lambda_2) \left(\frac{z}{f(z)} \right)^\alpha - \lambda_2 \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha \prec \frac{1 + A_1 z}{1 + B_1 z},$$

that is $f(z) \in N(\lambda_2, \alpha, A_1, B_1)$. So Theorem 2 is proved when $\lambda_2 = \lambda_1 \geq 0$.

When $\lambda_2 > \lambda_1 \geq 0$, it follows from Corollary 2 that $f(z) \in N(0, \alpha, A_1, B_1)$, that is

$$(3.3) \quad \left(\frac{z}{f(z)} \right)^\alpha \prec \frac{1 + A_1 z}{1 + B_1 z}.$$

But

$$(1 + \lambda_1) \left(\frac{z}{f(z)} \right)^\alpha - \lambda_1 \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha = \left(1 - \frac{\lambda_1}{\lambda_2} \right) \left(\frac{z}{f(z)} \right)^\alpha$$

$$+\frac{\lambda_1}{\lambda_2} \left[(1+\lambda_2) \left(\frac{z}{f(z)} \right)^\alpha - \lambda_2 \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha \right].$$

It is obvious that $h_1(z) = (1 + A_1z)/(1 + B_1z)$ is analytic and convex in U . So we obtain from Lemma 3 and differential subordinations (3.2) and (3.3) that

$$(1+\lambda_1) \left(\frac{z}{f(z)} \right)^\alpha - \lambda_1 \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha \prec \frac{1+A_1z}{1+B_1z},$$

that is $f(z) \in N(\lambda_1, \alpha, A_1, B_1)$. Thus we have

$$N(\lambda_2, \alpha, A_2, B_2) \subset N(\lambda_1, \alpha, A_1, B_1).$$

□

Corollary 3. Let $0 < \alpha < 1, \lambda_2 \geq \lambda_1 \geq 0, 1 > \beta_2 \geq \beta_1 \geq 0$, then

$$N(\lambda_2, \alpha, \beta_2) \subset N(\lambda_1, \alpha, \beta_1).$$

Theorem 3. Let $0 < \alpha < 1, \operatorname{Re} \lambda \geq 0, -1 \leq B < A \leq 1, f(z) \in N(\lambda, \alpha, A, B)$, then

$$(3.4) \quad \frac{\alpha}{\lambda} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\alpha}{\lambda}-1} du < \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} < \frac{\alpha}{\lambda} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\alpha}{\lambda}-1} du, \quad z \in U,$$

and inequality (3.4) is sharp, with the extremal function defined by

$$(3.5) \quad f_{\lambda, \alpha, A, B}(z) = z \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1+Az u}{1+Bz u} u^{\frac{\alpha}{\lambda}-1} du \right)^{-\frac{1}{\alpha}}.$$

Proof. Since $f(z) \in N(\lambda, \alpha, A, B)$, according to Theorem 1 we have

$$\left(\frac{z}{f(z)} \right)^\alpha \prec \frac{\alpha}{\lambda} \int_0^1 \frac{1+Az u}{1+Bz u} u^{\frac{\alpha}{\lambda}-1} du.$$

Therefore it follows from the definition of the subordination and $A > B$ that

$$\begin{aligned} \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} &< \sup_{z \in U} \operatorname{Re} \left\{ \frac{\alpha}{\lambda} \int_0^1 \frac{1+Az u}{1+Bz u} u^{\frac{\alpha}{\lambda}-1} du \right\} \\ &\leq \frac{\alpha}{\lambda} \int_0^1 \sup_{z \in U} \operatorname{Re} \left\{ \frac{1+Az u}{1+Bz u} \right\} u^{\frac{\alpha}{\lambda}-1} du \\ &< \frac{\alpha}{\lambda} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\alpha}{\lambda}-1} du; \\ \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} &> \inf_{z \in U} \operatorname{Re} \left\{ \frac{\alpha}{\lambda} \int_0^1 \frac{1+Az u}{1+Bz u} u^{\frac{\alpha}{\lambda}-1} du \right\} \\ &\geq \frac{\alpha}{\lambda} \int_0^1 \inf_{z \in U} \operatorname{Re} \left\{ \frac{1+Az u}{1+Bz u} \right\} u^{\frac{\alpha}{\lambda}-1} du \\ &> \frac{\alpha}{\lambda} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\alpha}{\lambda}-1} du. \end{aligned}$$

□

Note that $f_{\lambda, \alpha, A, B}(z) \in N(\lambda, \alpha, A, B)$, we obtain that inequality (3.4) is sharp. By applying the similar method as in Theorem 3, we have

Theorem 4. Let $0 < \alpha < 1, \operatorname{Re} \lambda \geq 0, -1 \leq A < B \leq 1, f(z) \in N(\lambda, \alpha, A, B)$, then

$$(3.6) \quad \frac{\alpha}{\lambda} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\alpha}{\lambda}-1} du < \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} < \frac{\alpha}{\lambda} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\alpha}{\lambda}-1} du, \quad z \in U,$$

and inequality (3.6) is sharp, with the extremal function defined by equation (3.5).

Corollary 4. Let $0 < \alpha < 1, \operatorname{Re} \lambda \geq 0, 0 \leq \beta < 1, f(z) \in N(\lambda, \alpha, \beta)$, then

$$(3.7) \quad \begin{aligned} \frac{\alpha}{\lambda} \int_0^1 \frac{1-(1-2\beta)u}{1+u} u^{\frac{\alpha}{\lambda}-1} du &< \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} \\ &< \frac{\alpha}{\lambda} \int_0^1 \frac{1+(1-2\beta)u}{1-u} u^{\frac{\alpha}{\lambda}-1} du, \quad z \in U, \end{aligned}$$

and inequality (3.7) is equivalent to

$$\begin{aligned} \beta + \frac{(1-\beta)\alpha}{\lambda} \int_0^1 \frac{1-u}{1+u} u^{\frac{\alpha}{\lambda}-1} du &< \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} \\ &< \beta + \frac{(1-\beta)\alpha}{\lambda} \int_0^1 \frac{1+u}{1-u} u^{\frac{\alpha}{\lambda}-1} du, \quad z \in U. \end{aligned}$$

Corollary 5. Let $0 < \alpha < 1, \operatorname{Re} \lambda \geq 0, \beta > 1, f(z) \in H$, and

$$\operatorname{Re} \left\{ (1+\lambda) \left(\frac{z}{f(z)} \right)^\alpha - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^\alpha \right\} < \beta, \quad z \in U,$$

then

$$(3.8) \quad \begin{aligned} \frac{\alpha}{\lambda} \int_0^1 \frac{1+(1-2\beta)u}{1-u} u^{\frac{\alpha}{\lambda}-1} du &< \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} \\ &< \frac{\alpha}{\lambda} \int_0^1 \frac{1-(1-2\beta)u}{1+u} u^{\frac{\alpha}{\lambda}-1} du, \quad z \in U, \end{aligned}$$

and inequality (3.8) is equivalent to

$$\begin{aligned} \beta + \frac{(1-\beta)\alpha}{\lambda} \int_0^1 \frac{1+u}{1-u} u^{\frac{\alpha}{\lambda}-1} du &< \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} \\ &< \beta + \frac{(1-\beta)\alpha}{\lambda} \int_0^1 \frac{1-u}{1+u} u^{\frac{\alpha}{\lambda}-1} du, \quad z \in U. \end{aligned}$$

If $\operatorname{Re} \omega \geq 0$, then $(\operatorname{Re} \omega)^{\frac{1}{2}} \leq \operatorname{Re} \omega^{\frac{1}{2}} \leq |\omega(z)|^{\frac{1}{2}}$ (see [2]), so we have

Theorem 5. Let $0 < \alpha < 1, \operatorname{Re} \lambda \geq 0, -1 \leq B < A \leq 1, f(z) \in N(\lambda, \alpha, A, B)$, then

$$(3.9) \quad \begin{aligned} \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\alpha}{\lambda}-1} du \right)^{\frac{1}{2}} &< \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^{\frac{\alpha}{2}} \right\} \\ &< \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\alpha}{\lambda}-1} du \right)^{\frac{1}{2}}, \quad z \in U, \end{aligned}$$

and inequality (3.9) is sharp, with the extremal function defined by equation (3.5).

Proof. According to Theorem 1 we have

$$\left(\frac{z}{f(z)} \right)^\alpha \prec \frac{1+Az}{1+Bz}.$$

Since $-1 \leq B < A \leq 1$, we have

$$0 \leq \frac{1-A}{1-B} < \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} < \frac{1+A}{1+B}.$$

Hence the result follows by Theorem 3. $\square$

Note that $f_{\lambda, \alpha, A, B}(z) \in N(\lambda, \alpha, A, B)$, we obtain that inequality (3.9) is sharp.

By applying the similar method as in Theorem 5, we have

Theorem 6. Let $0 < \alpha < 1$, $\operatorname{Re} \lambda \geq 0$, $-1 \leq A < B \leq 1$, $f(z) \in N(\lambda, \alpha, A, B)$, then

$$(3.10) \quad \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\alpha}{\lambda}-1} du \right)^{\frac{1}{2}} < \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^{\frac{\alpha}{2}} \right\} < \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\alpha}{\lambda}-1} du \right)^{\frac{1}{2}}, \quad z \in U,$$

and inequality (3.10) is sharp, with the extremal function defined by equation (3.5).

Theorem 7. Let $0 < \alpha < 1$, $\operatorname{Re} \lambda \geq 0$, $-1 \leq B < A \leq 1$, $f(z) \in N(\lambda, \alpha, A, B)$.

(i) If $\lambda = 0$, when $|z| = r < 1$, we have

$$(3.11) \quad r \left(\frac{1+Br}{1+Ar} \right)^{\frac{1}{\alpha}} \leq |f(z)| \leq r \left(\frac{1-Br}{1-Ar} \right)^{\frac{1}{\alpha}},$$

and inequality (3.11) is sharp, with the extremal function defined by

$$f(z) = z[(1+Bz)/(1-Az)]^{\frac{1}{\alpha}}.$$

(ii) If $\lambda \neq 0$, when $|z| = r < 1$, we have

$$(3.12) \quad r \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1+Bur}{1+Bur} u^{\frac{\alpha}{\lambda}-1} du \right)^{-\frac{1}{\alpha}} \leq |f(z)| \leq r \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1-Aur}{1-Bur} u^{\frac{\alpha}{\lambda}-1} du \right)^{-\frac{1}{\alpha}},$$

and inequality (3.12) is sharp, with the extremal function defined by equation (3.5).

Proof. (1) If $\lambda = 0$, since $f(z) \in N(\lambda, \alpha, A, B)$, $-1 \leq B < A \leq 1$, we obtain from the definition of $N(\lambda, \alpha, A, B)$ that

$$\left(\frac{z}{f(z)} \right)^\alpha \prec \frac{1+Az}{1+Bz}.$$

Therefore it follows from the definition of the subordination that

$$\left(\frac{z}{f(z)} \right)^\alpha = \frac{1+A\omega(z)}{1+B\omega(z)},$$

where $\omega(z)$ is analytic in U . By applying Schwarz Lemma we obtain that

$$\omega(z) = c_1 z + c_2 z^2 + \cdots$$

and $|\omega(z)| \leq |z|$, so when $|z| = r < 1$, we have

$$\left| \frac{z}{f(z)} \right|^\alpha = \left| \frac{1+A\omega(z)}{1+B\omega(z)} \right| \leq \frac{1+A|\omega(z)|}{1+B|\omega(z)|} \leq \frac{1+Ar}{1+Br},$$

and

$$\left| \frac{z}{f(z)} \right|^\alpha \geq \operatorname{Re} \left\{ \left(\frac{z}{f(z)} \right)^\alpha \right\} \geq \frac{1-Ar}{1-Br}.$$

It is obvious that inequality (3.11) is sharp, with the extremal function defined by $f(z) = z[(1+Bz)/(1+Az)]^{\frac{1}{\alpha}}$.

(2) If $\lambda \neq 0$, according to Theorem 1 we have

$$\left(\frac{z}{f(z)}\right)^\alpha \prec \frac{\alpha}{\lambda} \int_0^1 \frac{1+Az u}{1+Bzu} u^{\frac{\alpha}{\lambda}-1} du.$$

Therefore it follows from the definition of the subordination that

$$\left(\frac{z}{f(z)}\right)^\alpha = \frac{\alpha}{\lambda} \int_0^1 \frac{1+Au\omega(z)}{1+Bu\omega(z)} u^{\frac{\alpha}{\lambda}-1} du,$$

where $\omega(z) = c_1 z + c_2 z^2 + \dots$ is analytic in U and $|\omega(z)| \leq |z|$, so when $|z| = r < 1$, we have

$$\begin{aligned} \left|\frac{z}{f(z)}\right|^\alpha &\leq \frac{\alpha}{\lambda} \int_0^1 \left|\frac{1+Au\omega(z)}{1+Bu\omega(z)}\right| u^{\frac{\alpha}{\lambda}-1} du \leq \frac{\alpha}{\lambda} \int_0^1 \frac{1+Au|\omega(z)|}{1+Bu|\omega(z)|} u^{\frac{\alpha}{\lambda}-1} du \\ &\leq \frac{\alpha}{\lambda} \int_0^1 \frac{1+Aur}{1+Bur} u^{\frac{\alpha}{\lambda}-1} du, \end{aligned}$$

and

$$\left|\frac{z}{f(z)}\right|^\alpha \geq \operatorname{Re} \left\{ \left(\frac{z}{f(z)}\right)^\alpha \right\} \geq \frac{\alpha}{\lambda} \int_0^1 \frac{1-Aur}{1-Bur} u^{\frac{\alpha}{\lambda}-1} du.$$

□

Note that $f_{\lambda, \alpha, A, B}(z) \in N(\lambda, \alpha, A, B)$, we obtain that inequality (3.12) is sharp. By applying the similar method as in Theorem 7, we have

Theorem 8. Let $0 < \alpha < 1, \operatorname{Re} \lambda \geq 0, -1 \leq A < B \leq 1, f(z) \in N(\lambda, \alpha, A, B)$.

(i) If $\lambda = 0$, when $|z| = r < 1$, we have

$$(3.13) \quad r \left(\frac{1-Br}{1-Ar} \right)^{\frac{1}{\alpha}} \leq |f(z)| \leq r \left(\frac{1+Br}{1+Ar} \right)^{\frac{1}{\alpha}},$$

and inequality (3.13) is sharp, with the extremal function defined by

$$f(z) = z[(1+Bz)/(1+Az)]^{\frac{1}{\alpha}}.$$

(ii) If $\lambda \neq 0$, when $|z| = r < 1$, we have

$$r \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1-Aur}{1-Bur} u^{\frac{\alpha}{\lambda}-1} du \right)^{-\frac{1}{\alpha}} \leq |f(z)| \leq r \left(\frac{\alpha}{\lambda} \int_0^1 \frac{1+Aur}{1+Bur} u^{\frac{\alpha}{\lambda}-1} du \right)^{-\frac{1}{\alpha}}.$$

and inequality (3.14) is sharp, with the extremal function defined by equation (3.5).

Theorem 9. Let $0 < \alpha < 1, \lambda \in C, -1 \leq B \leq 1, A \neq B, A \in R$,

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in N(\lambda, \alpha, A, B),$$

then

$$|a_{n+1}| \leq \frac{|A-B|}{|\lambda n + \alpha|},$$

and inequality (3.15) is sharp, with the extremal function defined by equation (3.5).

Proof. Since $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in N(\lambda, \alpha, A, B)$, we have

$$(1 + \lambda) \left(\frac{z}{f(z)} \right)^{\alpha} - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^{\alpha} = 1 + (-\lambda n - \alpha) a_{n+1} z^n + \cdots \prec \frac{1 + Az}{1 + Bz}.$$

Hence it follows from Lemma 4 that $|(-\lambda n - \alpha) a_{n+1}| \leq |A - B|$, so

$$|a_{n+1}| \leq |A - B| / |\lambda n + \alpha|.$$

□

Note that $f(z) = z + [(A - B)/(\lambda n + \alpha)] z^{n+1} + \cdots \in N(\lambda, \alpha, A, B)$, we obtain that inequality (3.15) is sharp.

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INSTITUTE OF MATHEMATICS AND COMPUTING SCIENCE,
CHANGSHA UNIVERSITY OF SCIENCE AND TECHNOLOGY,
CHANGSHA, HUNAN 410076,
PEOPLE'S REPUBLIC OF CHINA
E-mail address: wzg429@tom.com