

APPROXIMATION PROPERTIES OF PARTIAL SUMS OF FOURIER SERIES

T. KARCHAVA

ABSTRACT. In this paper we find class of functions for which the Lebesgue estimate can be improved.

Let $C([0, 2\pi])$ denote the space of continuous function f with period 2π . If $f \in C([0, 2\pi])$ then the function

$$\omega_p(\delta, f) = \sup_x \sup_{|h| \leq \delta} |\Delta_p(x; h, f)|, \omega_1(\delta, f) = \omega(\delta, f)$$

is called the modulus of continuity of the function f , where

$$\Delta_1(x; h, f) = f(x + h) - f(x),$$

$$\Delta_{p+1}(x; h, f) = \Delta_p(x + h; h, f) - \Delta_p(x; h, f).$$

Denote by $Lip\alpha$ the class of function $f \in C([0, 2\pi])$ for which $\omega(\delta, f) \leq c(f) \delta^\alpha$ and let $S_n(f, x)$ be the n -th partial sum of the trigonometric Fourier series of the function f .

The estimation of Lebesgue (see [Dz, p. 116], or [Ba, Ch. 1]) is well known

$$\|f - S_n(f)\|_C \leq c\omega\left(\frac{1}{n}, f\right) \log(n+2).$$

Generalization of this estimation were studied by Chanturia [Ch], Oskolkov [Os], Karchava [Ka]. The questions devoted to estimation the uniform deviation of f from its partial Fourier sums with respect to the Walsh, Vilenkin (bounded and unbounded case) systems were discussed by Fine [Fi], Onnewer [On] Tevzadze [Te], Gát [Gá].

In this paper we consider the following characteristic of function f

$$\varphi_p(n, \delta; f) = \sup_x \sup_{|h| \leq \delta} \sum_{i=1}^n |\Delta_p(x_i; h, f)|,$$

where $x_i = x + (i-1)h$.

There arises a question: for what subclasses of classes of $C([0, 2\pi])$ the Lebesgue estimate can be improved?

We prove that the following are true

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Theorem 1. Let $f \in C([0, 2\pi])$. Then

$$\|f - S_n(f)\|_C \leq c(p) \sum_{k=1}^n \frac{\varphi_p(k; \pi/n, f)}{k^2}.$$

Corollary 1. Let the function f has a finite number of intervals of monotonicity on $[0, 2\pi]$ and $f \in \text{Lip } \alpha, 0 < \alpha < 1$. Then

$$\|f - S_n(f)\|_C \leq \frac{c(f, \alpha)}{n^\alpha}.$$

Corollary 2. Let the function f has a finite number of intervals of convexity or concavity on $[0, 2\pi]$, then

$$\|f - S_n(f)\|_C \leq c(f) \omega\left(\frac{1}{n}; f\right).$$

Corollary 3. Let the function f has a finite number of intervals of monotonicity on $[0, 2\pi]$. Then

$$\|f - S_n(f)\|_C \leq c(f) \frac{1}{n} \int_{1/n}^{\pi} \frac{\omega(t, f)}{t^2} dt.$$

Proof of Theorem 1. Let $T_n(x)$ be Vale-Poisson polynomial which provides best approximation of function f in the space $C([0, 2\pi])$ (see [Dz]), in particular

$$T_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) V_n(t) dt, \quad \frac{1}{\pi} \int_{-\pi}^{\pi} |V_n(t)| dt \leq 1.$$

Denote

$$g(t) := f(x+t) + f(x-t) - 2f(x) - T_n(x+t) - T_n(x-t) + 2T_n(x).$$

It evident that

$$\varphi_p(k, \delta, T_n) \leq c\varphi_p(k, \delta, f)$$

and

$$(1) \quad \sum_{i=k+1}^n \frac{(-1)^i}{i} = \frac{(-1)^{k+1}}{2k} + P_k + Q_n \text{ where } P_k \leq \frac{c}{k^2}, Q_n \leq \frac{c}{n}.$$

We write

$$(2) \quad \int_0^{\pi} g(t) D_n(t) dt = \int_0^{\pi} g(t) \left(D_n(t) - \frac{\sin nt}{t} \right) dt + \int_0^{\pi} g(t) \frac{\sin nt}{t} dt.$$

Since

$$D_n(t) - \frac{\sin nt}{t}$$

is bounded function and

$$|g(t)| \leq 4E_n(f),$$

we have

$$(3) \quad \left| \int_0^{\pi} g(t) \left(D_n(t) - \frac{\sin nt}{t} \right) dt \right| \leq cE_n(f).$$

where $E_n(f)$ is best approximation of function f .

On the other hand

$$(4) \quad \left| \int_0^{\pi/n} g(t) \frac{\sin nt}{t} dt \right| \leq \|g\|_C \int_0^{\pi/n} \frac{\sin nt}{t} dt \leq cE_n(f).$$

Combining (2)–(4) we have

$$(5) \quad \int_0^{\pi} g(t) D_n(t) dt = \int_0^{\pi} g(t) \frac{\sin nt}{t} dt + \gamma,$$

where

$$\gamma \leq cE_n(f).$$

It is evident that

$$(6) \quad \begin{aligned} \left| \int_{\pi/n}^{\pi} g(t) \frac{\sin nt}{t} dt \right| &= \left| \sum_{k=1}^{n-1} \int_{\pi k/n}^{\pi(k+1)/n} g(t) \frac{\sin nt}{t} dt \right| \\ &= \left| \sum_{k=1}^{n-1} \int_0^{\pi/n} g\left(t + \frac{\pi k}{n}\right) \frac{(-1)^k \sin nt}{t + \frac{\pi k}{n}} dt \right| \\ &= \left| \sum_{k=1}^{n-1} \int_0^{\pi} g\left(\frac{u + \pi k}{n}\right) \frac{(-1)^k \sin u}{u + \pi k} du \right| \\ &\leq \pi \left| \sum_{k=1}^{n-1} g\left(\frac{u_0 + \pi k}{n}\right) \frac{(-1)^k}{u + \pi k} \right| \\ &\leq \pi \left| \sum_{k=1}^{n-1} g\left(\frac{u_0 + \pi k}{n}\right) \left(\frac{(-1)^k}{u + \pi k} - \frac{(-1)^k}{\pi k} \right) \right| \\ &\quad + \pi \left| \sum_{k=1}^{n-1} g\left(\frac{u_0 + \pi k}{n}\right) \frac{(-1)^k}{\pi k} \right| \\ &= I + II, \end{aligned}$$

where

$$\sup_u \left| \sum_{k=1}^{n-1} g\left(\frac{u + \pi k}{n}\right) \frac{(-1)^k}{u + \pi k} \right| = \left| \sum_{k=1}^{n-1} g\left(\frac{u_0 + \pi k}{n}\right) \frac{(-1)^k}{u_0 + \pi k} \right|.$$

It is clear that

$$(7) \quad I \leq \|g\|_C \sum_{k=1}^{n-1} \frac{1}{k^2} = O(E_n(f)).$$

Denote

$$u_k := \frac{u_0 + \pi k}{n}.$$

Using Abel transformation and (1), for II we obtain

$$\begin{aligned}
II &= \left| \sum_{k=1}^{n-1} g(u_k) \frac{(-1)^k}{k} \right| \\
&\leq \left| \sum_{k=1}^{n-2} (g(u_{k+1}) - g(u_k)) \sum_{i=k+1}^{n-1} \frac{(-1)^i}{i} \right| \\
&\quad + \frac{1}{\pi} \left| g(u_1) \sum_{i=1}^{n-1} \frac{(-1)^i}{i} \right| \\
&\leq \left| \sum_{k=1}^{n-2} (g(u_{k+1}) - g(u_k)) \left(\frac{(-1)^{k+1}}{2k} + P_k + Q_n \right) \right| \\
&\quad + \left| g(u_1) \sum_{i=1}^{n-1} \frac{(-1)^i}{i} \right| \\
&\leq \left| \sum_{k=1}^{n-2} (g(u_{k+1}) - g(u_k)) \frac{(-1)^{k+1}}{2k} \right| + O(E_n(f)) \\
&= \left| \sum_{k=1}^{n-2} \Delta_1 \left(u_k; \frac{\pi}{n}, g \right) \frac{(-1)^{k+1}}{2k} \right| + O(E_n(f)).
\end{aligned}$$

Iterating this inequality we obtain

$$\begin{aligned}
II &\leq c(p) \left| \sum_{k=1}^{n-2} \Delta_p \left(u_k; \frac{\pi}{n}, g \right) \frac{(-1)^{k+1}}{2k} \right| + O(E_n(f)) \\
&\leq c(p) \sum_{k=1}^{n-2} \frac{|\Delta_p(u_k; \frac{\pi}{n}, g)|}{k} + O(E_n(f)).
\end{aligned}$$

Using Abel transformation we obtain

$$\begin{aligned}
\sum_{k=1}^{n-2} \frac{|\Delta_p(u_k; \frac{\pi}{n}, g)|}{k} &\leq \sum_{k=1}^{n-3} \frac{1}{k^2} \sum_{i=1}^k |\Delta_p(u_i; \frac{\pi}{n}, g)| \\
&\quad + \frac{1}{n-2} \sum_{k=1}^{n-2} |\Delta_p(u_k; \frac{\pi}{n}, g)|,
\end{aligned}$$

consequently,

$$\begin{aligned}
(8) \quad II &= O \left(\sum_{k=1}^n \frac{1}{k^2} \varphi_p \left(k, \frac{\pi}{n}, g \right) + O(E_n(f)) \right) \\
&= O \left(\sum_{k=1}^n \frac{1}{k^2} \varphi_p \left(k, \frac{\pi}{n}, f \right) \right).
\end{aligned}$$

Combining (5)–(8) we complete the proof of the Theorem 1. □

Proof of Corollary 1. Since

$$\varphi_1 \left(k, \frac{\pi}{n}; f \right) \leq c(f) \omega \left(\frac{\pi k}{n}; f \right) \leq c(f) \left(\frac{\pi k}{n} \right)^\alpha,$$

we have

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k^2} \varphi_1 \left(k, \frac{\pi}{n}; f \right) &\leq c(f) \sum_{k=1}^n \frac{\omega \left(\frac{\pi k}{n}; f \right)}{k^2} \\ &\leq \frac{c(f)}{n^\alpha} \sum_{k=1}^n \frac{1}{k^{2-\alpha}} \leq \frac{c(f, \alpha)}{n^\alpha}. \end{aligned}$$

□

Proof of Corollary 2. Since the function f has a finite number of intervals of convexity or concavity on $[0, 2\pi]$ we have

$$\varphi_2(k, \delta; f) = c(f) \omega(\delta, f),$$

consequently from Theorem 1 we get,

$$\|f - S_n(f)\|_C \leq c \sum_{k=1}^n \frac{1}{k^2} \varphi_2 \left(k, \frac{\pi}{n}; f \right) \leq c(f) \omega \left(\frac{1}{n}, f \right).$$

□

Proof of Corollary 3. From the fact that the function f has a finite number of intervals of monotonicity on $[0, 2\pi]$ we write

$$\varphi_1 \left(k, \frac{\pi}{n}; f \right) \leq c(f) \omega \left(\frac{\pi k}{n}; f \right),$$

then using Theorem 1 we complete the proof of Corollary 3 in the following way

$$\begin{aligned} \|f - S_n(f)\|_C &\leq c(f) \sum_{k=1}^n \frac{1}{k^2} \varphi_1 \left(k, \frac{\pi}{n}; f \right) \leq c(f) \sum_{k=1}^n \frac{\omega \left(\frac{\pi k}{n}; f \right)}{k^2} \\ &\leq c(f) \frac{1}{n} \sum_{k=1}^n \frac{\omega \left(\frac{k}{n}; f \right)}{(k/n)^2} \frac{1}{n} \leq c(f) \frac{1}{n} \int_{1/n}^{\pi} \frac{\omega(t, f)}{t^2} dt. \end{aligned}$$

□

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,
SOKHUMI BRANCH OF TBILISI STATE UNIVERSITY,
STR. JIKIA 10,
0128 TBILISI, GEORGIA