

ALMOST EVERYWHERE CONVERGENCE OF SUBSEQUENCE OF LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

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ABSTRACT. In this paper we prove that the maximal operator of the subsequence of logarithmic means of Walsh-Fourier series is weak type $(1,1)$. Moreover, the logarithmic means $t_{m_n}(f)$ of the function $f \in L$ converge a.e. to f as $n \rightarrow \infty$.

In the literature, it is known the notion of the Riesz's logarithmic means of a Fourier series. The n -th mean of the Fourier series of the integrable function f is defined by

$$\frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f)}{k}.$$

This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász, and Yabuta ([Sz], [Ya]). This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát ([14], [2]).

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{Q_n} \sum_{k=1}^{n-1} q_{n-k} S_k(f),$$

where $Q_n := \sum_{k=1}^{n-1} q_k$. If $q_k = \frac{1}{k}$, then we get the (Nörlund) logarithmic means:

$$\frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f)}{n-k}.$$

Móricz and Siddiqi [11] investigates the approximation properties of some special Nörlund means of Walsh-Fourier series of L^p functions in norm. The case, when $q_k = \frac{1}{k}$ is excluded, since the methods of Móricz are not applicable for logarithmic means. In [7] we proved some convergence and divergence properties of the logarithmic means of functions in the class of continuous functions, and in the Lebesgue space L . Among others, we proved that the maximal norm convergence function space of this logarithmic means is $L \log^+ L$. On the other hand, with respect to

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approximation properties of logarithmic means of multiple Walsh-Fourier series see for instance the papers ([6, 4, 5]).

In this paper we discuss a.e. convergence of subsequence of logarithmic means of Walsh-Fourier series of functions from the space $f \in L$. In particular, we prove that the maximal operator of the subsequence of logarithmic means of Walsh-Fourier series is weak type (1,1). Moreover, the logarithmic means $t_{m_n}(f)$ of the function $f \in L$ converge a.e. to f as $n \rightarrow \infty$. For this we apply some Gát idea from [1], [3].

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2) \\ -1, & \text{if } x \in [1/2, 1) \end{cases}, \quad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1 \text{ and } x \in [0, 1).$$

Let $w_0, w_1, \dots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if

$$k = 2^{n_1} + \dots + 2^{n_s}$$

is a positive integer with $n_1 > n_2 > \dots > n_s \geq 0$, then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [12].

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$(1) \quad D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 1/2^n) \\ 0, & \text{if } x \in [1/2^n, 1). \end{cases}$$

As usual, denote by $L(I)$ ($I := [0, 1)$) the set of all measurable functions defined on I , for which

$$\|f\|_1 = \int_0^1 |f(x)| dx < \infty.$$

The rectangular partial sums of Fourier series with respect to the Walsh system are defined by

$$S_n(f, x, y) = \sum_{m=0}^{n-1} \hat{f}(m) w_m(x),$$

where

$$\hat{f}(m) = \int_0^1 f(t) w_m(t) dt$$

is called the m -th Walsh-Fourier coefficient of function f .

The logarithmic means of Walsh-Fourier series is defined as follows

$$t_n(f, x) = \frac{1}{l_n} \sum_{i=1}^{n-1} \frac{S_i(f, x)}{n-i},$$

where

$$l_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is evident that

$$t_n(f, x, y) = \int_0^1 f(x \oplus t) F_n(t) dt,$$

where

$$F_n(t) = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k(t)}{n-k}$$

and $\oplus$ denotes the dyadic addition ([13, 9]).

For the maximal operator $t^*(f)$ we prove

Theorem 1. *Let $\{m_n : n \geq 1\}$ be sequence of positive integers for which*

$$\sum_{n=1}^{\infty} \frac{\log^2(m_n - 2^{\lfloor \log m_n \rfloor} + 1)}{\log m_n} < \infty.$$

Then the operator $t^(f) := \sup_{n \geq 1} |t_{m_n}(f)|$ is weak type $(1,1)$, i.e.*

$$\|t^*(f)\|_{weak-L} := \sup_{\lambda} \lambda \text{mes}(\{x : t^*(f, x) > \lambda\}) \leq c \|f\|_1.$$

Corollary 1. *Let $\{m_n : n \geq 1\}$ be from Theorem 1. and $f \in L(I)$. Then*

$$t_{m_n}(f, x) \rightarrow f(x) \text{ a.e. as } n \rightarrow \infty.$$

Corollary 2. *Let $f \in L(I)$. Then*

$$t_{2^n}(f, x) \rightarrow f(x) \text{ a.e. as } n \rightarrow \infty.$$

Following the works of Gát [1, 3] the base of the proof of Theorem 1. is the following lemma.

Lemma 1. *Let $\{m_n : n \geq 1\}$ be from Theorem 1. Then*

$$\int_{2^{-k}}^1 \sup_{n \geq n(k)} |F_{m_n}(x)| dx < \infty,$$

where $n(k) = \min\{n : \lfloor \log m_n \rfloor \geq k\}$.

In order to prove Lemma 1., we shall need the following Lemmas.

Lemma 2 ([8]). *Let $1 \leq j < 2^n$. Then*

$$D_{2^n-j}(u) = D_{2^n}(u) - w_{2^n-1}(u) D_j(u).$$

Let us denote by K_j the j th Fejér kernel function, that is, $K_j = \frac{1}{j} \sum_{i=1}^j D_i$.

Lemma 3. *We have*

$$\int_{2^{-k}}^1 \sup_{n \geq 2^k} |K_n(x)| dx < \infty.$$

The proof can be found in work of Gát [1].

Lemma 4. *Let $2^n \leq m < 2^{m+1}$. Then*

$$\begin{aligned} l_m F_m(x) &= l_m D_{2^n}(x) \\ &\quad - w_{2^n-1}(x) \sum_{j=1}^{2^n-2} \left(\frac{1}{m-2^n+j} - \frac{1}{m-2^n+j+1} \right) j K_j(x) \\ &\quad - \frac{2^n-1}{m-1} w_{2^n-1}(x) K_{2^n-1}(x) + w_{2^n}(x) l_{m-2^n} F_{m-2^n}(x). \end{aligned}$$

Proof of Lemma 4. It is evident that

$$(2) \quad l_m F_m(x) = \sum_{j=1}^{2^n} \frac{D_j(x)}{m-j} + \sum_{j=2^n+1}^{m-1} \frac{D_j(x)}{m-j} = I + II.$$

Using Abel transformation and Lemma 1. we have

$$\begin{aligned} I &= \sum_{j=0}^{2^n-1} \frac{D_{2^n-j}(x)}{m-2^n+j} = \frac{D_{2^n}(x)}{m-2^n} + \sum_{j=1}^{2^n-1} \frac{D_{2^n-j}(x)}{m-2^n+j} \\ &= \frac{D_{2^n}(x)}{m-2^n} + D_{2^n}(x) \left(\sum_{j=1}^{2^n-1} \frac{1}{m-2^n+j} \right) \\ (3) \quad &\quad - w_{2^n-1}(x) \sum_{j=1}^{2^n-1} \frac{D_j(x)}{m-2^n+j} = (l_m - l_{m-2^n}) D_{2^n}(x) \\ &\quad - w_{2^n-1}(x) \sum_{j=1}^{2^n-2} \left(\frac{1}{m-2^n+j} - \frac{1}{m-2^n+j+1} \right) j K_j(x) \\ &\quad - \frac{2^n-1}{m-1} w_{2^n-1}(x) K_{2^n-1}(x). \end{aligned}$$

Since

$$D_{j+2^n}(x) = D_{2^n}(x) + w_{2^n}(x) D_j(x), \quad j = 1, 2, \dots, 2^n-1,$$

for II we write

$$(4) \quad II = \sum_{j=1}^{m-2^n-1} \frac{D_{j+2^n}(x)}{m-2^n-j} = l_{m-2^n} D_{2^n}(x) + w_{2^n}(x) l_{m-2^n} F_{m-2^n}(x).$$

Combining (2)–(4) we complete the proof of Lemma 4. □

Lemma 5. *Let $\lim_{n \rightarrow \infty} \frac{\log^2(m_n - 2^{\lfloor \log m_n \rfloor + 1})}{\log m_n} < \infty$. Then*

$$\|F_{m_n}\|_1 \leq c < \infty, \quad n = 1, 2, \dots$$

Proof of Lemma 5. Since

$$\|F_n\|_1 \leq \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\|D_j\|_1}{n-j} \leq \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{\ln(j+1)}{n-j} = O(l_n)$$

and

$$(5) \quad \|K_n\|_1 \leq c < \infty, \quad n = 1, 2, \dots$$

from Lemma 4. we have

$$\begin{aligned} \|F_{m_n}\|_1 &\leq 1 + \frac{1}{l_{m_n}} \sum_{j=1}^{2^{\lfloor \log m_n \rfloor - 2}} \frac{\|K_j\|_1}{j} \\ &\quad + \|K_{2^{\lfloor \log m_n \rfloor - 1}}\|_1 + \frac{l_{m_n - 2^{\lfloor \log m_n \rfloor}}}{l_{m_n}} \|F_{m_n - 2^{\lfloor \log m_n \rfloor}}\|_1 \\ &= O\left(\frac{\log^2(m_n - 2^{\lfloor \log m_n \rfloor} + 1)}{\log m_n}\right) = O(1). \end{aligned}$$

Lemma 5. is proved. $\square$

Proof of Lemma 1. From Lemma 3. and by (1), (5) we have

$$\begin{aligned} \int_{2^{-k}}^1 \sup_{n \geq n(k)} |F_{m_n}(x)| dx &\leq c_1 \int_{2^{-k}}^1 \sup_{n \geq n(k)} \frac{1}{\log m_n} \sum_{j=1}^{2^{\lfloor \log m_n \rfloor - 2}} \frac{|K_j(x)|}{j} dx \\ &\quad + c_2 \int_{2^{-k}}^1 \sup_{n \geq n(k)} |K_{2^{\lfloor \log m_n \rfloor - 1}}(x)| dx \\ &\quad + \int_{2^{-k}}^1 \sup_{n \geq n(k)} \frac{\log(m_n - 2^{\lfloor \log m_n \rfloor} + 1)}{\log m_n} |F_{m_n - 2^{\lfloor \log m_n \rfloor}}(x)| dx \\ &\leq c_3 + c_1 \int_{2^{-k}}^1 \sup_{n \geq n(k)} \frac{1}{\log m_n} \sum_{j=1}^{2^k - 1} \frac{|K_j(x)|}{j} dx \\ &\quad + c_1 \int_{2^{-k}}^1 \sup_{n \geq n(k)} \frac{1}{\log m_n} \sum_{j=2^k}^{2^{\lfloor \log m_n \rfloor - 2}} \frac{1}{j} \sup_{i \geq 2^k} |K_i(x)| dx \\ &\quad + \sum_{n=1}^{\infty} \frac{\log(m_n - 2^{\lfloor \log m_n \rfloor} + 1)}{\log m_n} \int_0^1 |F_{m_n - 2^{\lfloor \log m_n \rfloor}}(x)| dx \\ &\leq c_4 + c_5 \int_{2^{-k}}^1 \sup_{i \geq 2^k} |K_i(x)| dx + \sum_{n=1}^{\infty} \frac{\log^2(m_n - 2^{\lfloor \log m_n \rfloor} + 1)}{\log m_n} \leq c_6 < \infty. \end{aligned}$$

Lemma 1. is proved. $\square$

Given $u \in I$, let $I_k(u)$ denote a dyadic interval of length 2^{-k} which contains the point u .

In the sequel we prove that the maximal operator $t^*(f)$ is quasi-local. This reads as follows

Lemma 6. *Let $f \in L(I)$, $\text{supp } f \subset I_k(u)$ and $\int_{I_k(u)} f(x) dx = 0$ for some $u \in I$.*

Then

$$\int_{2^{-k}}^1 t^*(f, x) dx \leq c \|f\|_1.$$

Proof of Lemma 6. By the shift invariency of the Haar measure it can be supposed that $u = 0$. If $n \leq n(k)$ then

$$\begin{aligned} t_{m_n}(f, x) &= \int_I f(u) F_{m_n}(x \oplus u) du \\ &= F_{m_n}(x) \int_0^{2^{-k}} f(u) du = 0. \end{aligned}$$

Consequently, $n > n(k)$ can be supposed.

Then from Lemma 1 we have

$$\begin{aligned} \int_{2^{-k}}^1 t^*(f, x) dx &\leq \int_0^{2^{-k}} |f(u)| \left(\int_{2^{-k}}^1 \sup_{n \geq n(k)} |F_{m_n}(x \oplus u)| dx \right) du \\ &\leq c \|f\|_1. \end{aligned}$$

□

Proof of Theorem 1. As a consequence of Lemma 5. we have that the maximal operator $t^*(f)$ is of type (∞, ∞) . Since the sublinear operator is quasi-local, then by standard argument [13] it follows that it is of weak type $(1, 1)$. □

By making use of the well-known density argument due to Marcinkiewicz and Zygmund [10] we can show that Corollary 1. follows from Theorem 1.

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