Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
21 (2005), 177–183
`www.emis.de/journals`
`ISSN 1786-0091`

ON THE POINTWISE ESTIMATION OF CESARO KERNEL OF NEGATIVE ORDER WITH RESPECT TO WALSH-PALEY SYSTEM

V. TEVZADZE

ABSTRACT. Some pointwise properties of (C,α) kernel $(-1<\alpha<0)$ with respect to the Walsh–Paley system are established.

1. INTRODUCTION

Let $r_0(x)$ be the function defined by

$$r_0(x)=\begin{cases}1 & \text{if } x\in\left[0,\frac12\right),\\ -1 & \text{if } x\in\left[\frac12,1\right),\end{cases}\qquad r_0(x+1)=r_0(x).$$

The Rademacher system is defined by

$$r_n(x)=r_0(2^nx),\quad n\ge 1,\ \text{and } x\in[0,1).$$

Let $\psi_0(x),\psi_1(x),\psi_2(x),\dots$ represent the Walsh functions, i.e. $\psi_0(x)=1$, and if $k=2^{n_1}+2^{n_2}+\dots+2^{n_s}$ is a positive integer with $n_1>n_2>\dots>n_s$, then

$$\psi_k(x)=r_{n_1}(x)\cdot r_{n_2}(x)\cdots r_{n_s}(x).$$

The idea of using the products of the Rademacher functions is to define the Walsh system originated by Paley [4].

Denote by $K_n^\alpha(t)$ the kernel of the method (C,α) and call it the (C,α) kernel, or the Cesaro kernel:

$$K_n^\alpha(t)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^\alpha\psi_\nu(t),$$

$$A_k^\alpha=\frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!}\quad(\alpha\ne -k).$$

It is well-known ([8], Ch. 3) that

(1) $A_n^\alpha=\sum_{k=0}^{n}A_{n-k}^{\alpha-1}$;
(2) $A_n^\alpha-A_{n-1}^\alpha=A_n^{\alpha-1}$;
(3) $A_n^\alpha\sim n^\alpha$.

2000 *Mathematics Subject Classification.* 42C10.
Key words and phrases. Walsh functions, Dirichlet kernel, Cesaro means.

Some properties of the (C,α) kernel $(\alpha>0)$ have been established by Fine [1], Yano [7], Gát [2], [3] and [5]. Using these properties they studied the problems of pointwise and uniform (C,α) summability of Walsh–Fourier series.

In the present paper we study some pointwise properties of (C,α) kernel $(-1<\alpha<0)$ with respect to the Walsh–Paley system. The results of this paper have been published without proof in [6].

2. Main Results

The main results of the paper are presented in the form of the following propositions.

Theorem 1. *The estimation*

$$K_n^{-\alpha}(t)\le c(\alpha)\cdot\frac{1}{A_n^{-\alpha}}\cdot\frac{1}{t^{1-\alpha}},\quad t\in(0,1),\quad 0<\alpha<1,$$

holds.

Theorem 2. *For any $\alpha\in(0,1)$ and $p\ge 2^m$ the equality*

$$\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^m-1}A_{p-\nu}^{-\alpha}\psi_\nu(t)\Big)=\operatorname{Sgn}\psi_{2^m-1}(t),\quad t\in[0,1),$$

is valid.

3. Auxiliary Results

We shall need the following

Lemma 1. *For any $\alpha>0$ and $p>2^m-1+\alpha$ the sum*

$$\sum_{\nu=0}^{2^m-1}A_{p-\nu}^{-\alpha}\psi_\nu(t)$$

is representable in the form

$$\sum_{\nu=0}^{2^m-1}A_{p-\nu}^{-\alpha}\psi_\nu(t)=\sum_{\nu=0}^{2^m-1}\ell_\nu A_{p-q_\nu}^{-\alpha-i},\tag{1}$$

where ℓ_ν,q_ν,i are nonnegative integers depending only on the point $t\in[0,1]$ and $m\in\mathbb{N}$ (and not depending on p and α); moreover, $i+q_\nu\le 2^m-1$.

Proof. Using the method of mathematical induction, we can verify that the lemma is valid for $m=1$. We have

$$\sum_{\nu=0}^{2^1-1}A_{p-\nu}^{-\alpha}\psi_\nu(t)=A_p^{-\alpha}\psi_0(t)+A_{p-1}^{-\alpha}\psi_1(t).$$

Since on the interval $0\le t<\frac12$

$$A_p^{-\alpha}\psi_0(t)+A_{p-1}^{-\alpha}\psi_1(t)=A_p^{-\alpha}+A_{p-1}^{-\alpha},$$

and on the interval $\frac12\le t<1$

$$A_p^{-\alpha}\psi_0(t)+A_{p-1}^{-\alpha}\psi_1(t)=A_p^{-\alpha}-A_{p-1}^{-\alpha}=A_p^{-\alpha-1},$$

our lemma for $m = 1$ is valid. Let the lemma be valid for $m-1 \in \mathbb{N}$, and we prove that the lemma is valid for $m \in \mathbb{N}$. Indeed, we have

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) + \psi_{2^{m-1}}(t) \sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t)$$

$$(\alpha > 0, \quad p > 2^m - 1 + \alpha).$$

We consider two cases: (1) $\psi_{2^{m-1}}(t) = 1$; (2) $\psi_{2^{m-1}}(t) = -1$.

Let $\psi_{2^{m-1}}(t) = 1$. By the assumption

$$p - 2^{m-1} > 2^m - 1 + \alpha - 2^{m-1} > 2^{m-1} - 1 + \alpha,$$

$$\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-m_\nu}^{-\alpha-i}$$

and

$$\sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-2^{m-1}-m_\nu}^{-\alpha-i}.$$

Hence we have

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-m_\nu}^{-\alpha-i} + \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-2^{m-1}-m_\nu}^{-\alpha-i} = \sum_{\nu=0}^{2^m-1} \ell_\nu A_{p-q_\nu}^{-\alpha-i}.$$

Let now $\psi_{2^{m-1}} = -1$. Since $A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$, we have

$$\begin{aligned}\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) &= \sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) + \psi_{2^{m-1}}(t) \sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t) \\ &= \sum_{\nu=0}^{2^{m-1}-1} \left(A_{p-\nu}^{-\alpha} - A_{p-2^{m-1}-\nu}^{-\alpha}\right)\psi_\nu(t) \qquad (2) \\ &= \sum_{\nu=0}^{2^{m-1}-1} \left(A_{p-\nu}^{-\alpha-1} + A_{p-\nu-1}^{-\alpha-1} + \cdots + A_{p-2^{m-1}-\nu-1}^{-\alpha-1}\right)\psi_\nu(t).\end{aligned}$$

By the assumption,

$$\sum_{\nu=0}^{2^{m-1}-1} A_{p-j-\nu}^{-\alpha-1} = \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-j-m_\nu}^{-\alpha-1-i}, \quad j = 0, 1, 2, \ldots, 2^{m-1}-1,$$

hence from (2) we have

$$\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha-1}\psi_\nu(t) = \sum_{j=0}^{2^{m-1}-1} \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-j-m_\nu}^{-\alpha-1-i} = \sum_{\nu=0}^{2^m-1} \ell_\nu A_{p-q_\nu}^{-\alpha-1-i},$$

i.e. in both cases equation (1) holds. It is evident that in these cases

$$i + q_\nu \le 2^m - 1.$$

Thus the lemma is proved. □

Lemma 2. *For any $\alpha > 0$ and $p > 2^m + \alpha$ the equality*

$$\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) = -\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha-1}\psi_\nu(t)\right), \quad t \in [0, 1),$$

is valid.

Lemma 2 follows directly from Lemma 1 if we take into account that in the conditions of Lemma 2 $\operatorname{Sgn} A^{-\alpha}_{p-\nu} = -\operatorname{Sgn} A^{-\alpha-1}_{p-\nu}$.

4. Proof of Main Results

Proof of Theorem 1. Let $t \in (0,1)$ and $m \in \mathbb{N}$ ($\mathbb{N}$ is a set of natural numbers), such that $2^{-m} \le t < 2^{-m+1}$. We write $n \ge 1$ in the form $n = p \cdot 2^m + q$, where $0 \le q < 2^m$. We have[1]

$$
\begin{aligned}
K^{-\alpha}_n(t) &= \frac{1}{A^{-\alpha}_n} \sum_{\nu=0}^{n} A^{-\alpha}_{n-\nu} \psi_\nu(t) = \frac{1}{A^{-\alpha}_n} \sum_{\nu=0}^{p\cdot 2^m-1} A^{-\alpha}_{n-\nu} \psi_\nu(t) \\
&\quad + \frac{1}{A^{-\alpha}_n} \sum_{\nu=p\cdot 2^m}^{n} A^{-\alpha}_{n-\nu} \psi_\nu(t) \\
&= \frac{1}{A^{-\alpha}_n} \sum_{r=0}^{p-1} \sum_{\nu=0}^{2^m-1} A^{-\alpha}_{n-r\cdot 2^m-\nu} \psi_{r\cdot 2^m+\nu}(t) \\
&\quad + \frac{1}{A^{-\alpha}_n} \sum_{\nu=0}^{q} A^{-\alpha}_{n-p\cdot 2^m-\nu} \psi_{p\cdot 2^m+\nu}(t) \\
&= \frac{1}{A^{-\alpha}_n} \sum_{r=0}^{p-2} \psi_{r\cdot 2^m} \sum_{\nu=0}^{2^m-1} A^{-\alpha}_{n-r\cdot 2^m-\nu} \psi_\nu(t) \\
&\quad + \frac{1}{A^{-\alpha}_n} \psi_{(p-1)\cdot 2^m}(t) \sum_{\nu=0}^{2^m-1} A^{-\alpha}_{n-r\cdot 2^m-\nu} \psi_\nu(t) \\
&\quad + \frac{1}{A^{-\alpha}_n} \psi_{p\cdot 2^m}(t) \sum_{\nu=0}^{q} A^{-\alpha}_{q-\nu} \psi_\nu(t) = A_1 + A_2 + A_3.
\end{aligned} \tag{3}
$$

Estimate A_1. Using Abelian transformation, we have

$$
\begin{aligned}
A_1 &= \frac{1}{A^{-\alpha}_n} \left| \sum_{r=0}^{p-2} \psi_{r\cdot 2^m}(t) \sum_{\nu=0}^{2^m-1} A^{-\alpha}_{n-r2^m-\nu} \psi_\nu(t) \right| \\
&= \frac{1}{A^{-\alpha}_n} \left| \sum_{r=0}^{p-2} \psi_{r\cdot 2^m}(t) \sum_{\nu=0}^{2^m-2} A^{-\alpha-1}_{n-r\cdot 2^m-\nu} D_\nu(t) \right. \\
&\quad \left. + \sum_{r=0}^{p-2} \psi_{r\cdot 2^m} A^{-\alpha}_{n-(r+1)2^m+1} D_{2^m}(t) \right|,
\end{aligned}
$$

where

$$
D_k(t) = \sum_{i=0}^{k-1} \psi_k(t).
$$

Since (see [8])

$$
c_1(\alpha) n^\alpha \le A^\alpha_n \le c_2(\alpha) n^\alpha, \quad \alpha > -2, \tag{4}
$$

---

[1] Here the use is made of the equality $\psi_{r+s}(t) = \psi_r(t)\psi_s(t)$, if in the binary expansion $r, s \in \mathbb{N}$ the same terms are omitted.

and $|D_k(t) \leq k$, $t \in [0,1]$, we obtain

$$\begin{aligned}|A_1| &\leq \frac{1}{A_n^{-\alpha}} \cdot c(\alpha) \cdot 2^m \sum_{r=0}^{p-2} \sum_{\nu=0}^{2^m-1} (n - r \cdot 2^m - \nu)^{-\alpha-1} \\ &\leq \frac{1}{A_n^{-\alpha}} \cdot c(\alpha) \cdot 2^m (n - (p-1) \cdot 2^m)^{-\alpha} \leq \frac{1}{A_n^{-\alpha}} \cdot c(\alpha) 2^{m(1-\alpha)} \\ &\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \cdot \frac{1}{t^{1-\alpha}}.\end{aligned} \tag{5}$$

For A_2 we have

$$\begin{aligned}|A_2| &= \frac{1}{A_n^{-\alpha}} \left| \psi_{(p-1)2^m}(t) \sum_{\nu=0}^{2^m-1} A_{n-(p-1)2^m-\nu}^{-\alpha} \psi_\nu(t) \right| \\ &\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{2^m-1} (n - (p-1)2^m - \nu)^{-\alpha} \\ &\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{2^m-1} (2^m + q - \nu)^{-\alpha} \\ &\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{2^m-1} (2^m - \nu)^{-\alpha} \leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} 2^{m(1-\alpha)} \\ &\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \cdot \frac{1}{t^{1-\alpha}}.\end{aligned} \tag{6}$$

Estimate now A_3,

$$\begin{aligned}|A_3| &= \frac{1}{A_n^{-\alpha}} \left| \sum_{\nu=0}^{q} A_{q-\nu}^{-\alpha} \psi_\nu(t) \right| \leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \left(1 + \sum_{\nu=0}^{q} (q-\nu)^{-al} \right) \\ &\leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \, q^{1-\alpha} \leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \, 2^{m(1-\alpha)} \leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \, \frac{1}{t^{1-\alpha}}.\end{aligned} \tag{7}$$

Taking into account (5), (6) and (7), from (3) we get

$$|K_n^{-\alpha}| \leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \, \frac{1}{t^{1-\alpha}},$$

which was to be proved. □

Proof of Theorem 2. We use the method of mathematical induction. For $m = 1$ the lemma is valid. Indeed,

$$\sum_{\nu=0}^{2^1-1} A_{p-\nu}^{-\alpha} \psi_\nu(t) = A_p^{-\alpha} \psi_0(t) + A_{p-1}^{-\alpha} \psi_1(t);$$

on the interval $0 \leq t < \frac{1}{2}$

$$A_p^{-\alpha} \psi_0(t) + A_{p-1}^{-\alpha} \psi_1(t) = A_p^{-\alpha} + A_{p-1}^{-\alpha} > 0,$$

and on the interval $\frac{1}{2} \leq t < 1$

$$A_p^{-\alpha} \psi_0(t) + A_{p-1}^{-\alpha} \psi_1(t) = A_p^{-\alpha} - A_{p-1}^{-\alpha} = A_p^{-\alpha-1} < 0.$$

Since $\psi_1(t)=1$ if $0\le t<\frac{1}{2}$, and $\psi_1(t)=-1$ if $\frac{1}{2}\le t<1$, therefore

$$\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^1-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\Big)=\operatorname{Sgn}\psi_1(t).$$

Let the lemma be valid for $m-1\in\mathbb{N}$ and let us prove that the lemma is valid for $m\in\mathbb{N}$. We have

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)=\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)+\psi_{2^{m-1}}(t)\sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t). \tag{8}$$

Let $\psi_{2^{m-1}}=1$. Since $p-2^{m-1}\ge 2^{m-1}$, by virtue of the assumption,

$$\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\Big)=\operatorname{Sgn}\psi_{2^{m-1}-1}(t),$$

$$\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t)\Big)=\operatorname{Sgn}\psi_{2^{m-1}-1}(t).$$

Hence from (8) it follows that

$$\begin{aligned}\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\Big)&=\operatorname{Sgn}\psi_{2^{m-1}-1}(t)\\&=\operatorname{Sgn}\psi_{2^{m-1}-1}(t)\cdot\operatorname{Sgn}\psi_{2^{m-1}}(t)\\&=\operatorname{Sgn}\big(\psi_{2^{m-1}-1}(t)\psi_{2^{m-1}}(t)\big)=\operatorname{Sgn}\psi_{2^m-1}(t).\end{aligned}$$

If, however, $\psi_{2^{m-1}}(t)=-1$, equality (8) yields

$$\begin{aligned}\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)&=\sum_{\nu=0}^{2^{m-1}-1}\Big(A_{p-\nu}^{-\alpha}-A_{p-2^{m-1}-\nu}^{-\alpha}\Big)\psi_\nu(t)\\&=\sum_{\nu=0}^{2^{m-1}-1}\Big(A_{p-\nu}^{-\alpha-1}+A_{p-\nu-1}^{-\alpha-1}+\cdots+A_{p-2^{m-1}-\nu+1}^{-\alpha-1}\Big)\psi_\nu(t).\end{aligned} \tag{9}$$

Taking into account Lemma 1, for all $j=1,2,\dots,2^{m-1}-1$ we obtain

$$\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\Big)=\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^{m-1}-1} A_{p-j-\nu}^{-\alpha-1}\psi_\nu(t)\Big),$$

and consequently, using Lemma 2, from (9) it follows that

$$\begin{aligned}\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\Big)&=\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha-1}\psi_\nu(t)\Big)\\&=-\operatorname{Sgn}\Big(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\Big)=-\operatorname{Sgn}\psi_{2^{m-1}-1}(t)\\&=\operatorname{Sgn}\psi_{2^{m-1}}(t)\operatorname{Sgn}\psi_{2^{m-1}-1}(t)=\operatorname{Sgn}\psi_{2^m-1}(t).\end{aligned}$$

Thus the theorem is proved. □

References

[1] N. J. Fine. Cesàro summability of Walsh-Fourier series. *Proc. Nat. Acad. Sci. U.S.A.*, 41:588–591, 1955.

[2] Gát, G. On the Fejr kernel functions with respect to the Walsh-Paley system. *Acta Acad. Paed. Agriensis Sectio Matematicae*, 24:105–110, 1997.

[3] B. Golubov, A. Efimov, and V. Skvortsov. *Walsh series and transforms*, volume 64 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Theory and applications, Translated from the 1987 Russian original by W. R. Wade.

[4] R. Paley. A remarkable series of orthogonal functions. I., II. *Proc. Lond. Math. Soc., II. Ser.*, 34:241–264, 265–279, 1932.

[5] F. Schipp, W. R. Wade, and P. Simon. *Walsh series.* Adam Hilger Ltd., Bristol, 1990. An introduction to dyadic harmonic analysis, With the collaboration of J. Pál.

[6] V. I. Tevzadze. Uniform convergence of Cesàro means of negative order of Fourier-Walsh series. *Soobshch. Akad. Nauk Gruzin. SSR*, 102(1):33–36, 1981.

[7] S. Yano. Cesàro summability of Fourier series. *J. Math. Tokyo*, 1:32–34, 1951.

[8] A. Zygmund. *Trigonometric series. 2nd ed. Vol. I.* Cambridge University Press, New York, 1959.

Received April 15, 2005.

Department of Mechanics and Mathematics,
Tbilisi State University,
Chavchavadze str. 1, Tbilisi 0128,
Georgia