

ABSOLUTE CONVERGENCE OF THE DOUBLE SERIES OF FOURIER – HAAR COEFFICIENTS

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ABSTRACT. In this paper we study the absolute convergence of the double series of Fourier-Haar coefficients of the class PBV_p .

1. INTRODUCTION

The problems related to the behaviour of single series of Fourier-Haar are well studied [9]. Namely, P. Ulianov [14] and B. Golubov [8] received the results related to the problems of absolute convergence of the series of Fourier-Haar coefficients. Some generalization of these results related were received by Z. Chanturia [3], T. Akhobadze [1], U. Goginava [7] and by the author [2]. In the term of modulus of smoothness the problem of absolute convergence of the series of Fourier-Haar coefficients was studied by V. Krotov [10]. Multidimensional analogies corresponding to the results of V. Krotov were formulated in the works of V. Tsagareishvili [13] and G. Tabatadze [12].

The estimates of Fourier coefficients of functions of bounded fluctuation with respect to Walsh system were studied in [11] and with respect to Vilenkin system were studied by G. Gát and R. Toledo [4].

We consider the double Haar system $\{\chi_n(x) \times \chi_m(y) : n, m = 0, 1, 2, \dots\}$ on the unit square $I^2 = [0, 1] \times [0, 1]$. As usual, $L_p(I^2)$ ($p \geq 1$) denotes the set of all measurable functions defined on I^2 , for which

$$\|f\|_p = \left(\int_0^1 \int_0^1 |f(x, y)|^p dx dy \right)^{\frac{1}{p}} < \infty$$

and $C(I^2)$ is the space of continuous functions on I^2 equipped with maximum norm

$$\|f\|_c = \max_{x, y \in I} |f(x, y)|.$$

If $f \in L(I^2)$, then

$$C_{n,m}(f) = \int_0^1 \int_0^1 f(x, y) \chi_n(x) \chi_m(y) dx dy$$

is the (n, m) th Fourier-Haar coefficient of f .

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We say that $f \in \text{Lip } \alpha$ on $[0, 1]^2$, if

$$\|f(\cdot + h, \cdot + \eta) - f(\cdot, \cdot)\|_c = O\left((h^2 + \eta^2)^{\frac{\alpha}{2}}\right), \alpha \in (0, 1].$$

We have the following theorem.

Theorem A ([12]). *a) Let $f \in \text{Lip } \alpha$ on $[0, 1]^2$, $\alpha \in (0, 1]$. If $\beta > 0$ and $\gamma + 1 < \beta \frac{(\alpha+1)}{2}$, then*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (nm)^{\gamma} |C_{n,m}(f)|^{\beta} < \infty.$$

b) Let $\gamma + 1 = \beta \frac{(\alpha+1)}{2}$, for some $\alpha \in (0, 1)$. Then there exists a function $f_{\alpha} \in \text{Lip } \alpha$ for which

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (nm)^{\gamma} |C_{n,m}(f_{\alpha})|^{\beta} = \infty.$$

The case for $\gamma = 0$ was considered earlier by V. Tsagareishvili [13].

Let $f \in L_p(I^2)$. The partial integrated modulus of continuity are defined by

$$\begin{aligned} \omega_1(\delta_1, f)_p &= \sup \left\{ \|f(x+u, y) - f(x, y)\|_p : |u| \leq \delta_1 \right\}, \\ \omega_2(\delta_2, f)_p &= \sup \left\{ \|f(x, y+v) - f(x, y)\|_p : |v| \leq \delta_2 \right\}. \end{aligned}$$

We also use the notion of the mixed integrated modulus of continuity. It is defined as follows

$$\begin{aligned} \omega_{1,2}(\delta_1, \delta_2, f)_p &= \sup \left\{ \|f(x+u, y+v) - f(x+u, y) - f(x, y+v) + f(x, y)\|_p : \right. \\ &\quad \left. |u| \leq \delta_1, |v| \leq \delta_2 \right\}, f \in L_p(I^2). \end{aligned}$$

It is not difficult to show that

$$(1) \quad \omega_{1,2}(\delta_1, \delta_2, f)_p \leq 2\sqrt{\omega_1(\delta_1, f)_p} \sqrt{\omega_2(\delta_2, f)_p}.$$

We study the problem of absolute convergence of the series of Fourier-Haar coefficients for the classes of functions with bounded partial p-variations, which were first considered by U. Goginava (see [5] for $p = 1$ and [6] for $p > 1$).

Definition. A function $f: I^2 \rightarrow R$ is said to be of bounded partial p-variation ($f \in \text{PBV}_p(I^2)$) if there exists a constant K such that for any partition

$$\begin{aligned} \Delta_1: 0 \leq x_0 < x_1 < x_2 < \dots < x_n \leq 1, \\ \Delta_2: 0 \leq y_0 < y_1 < y_2 < \dots < y_m \leq 1, \end{aligned}$$

we have

$$\begin{aligned} V_1(f)_p &= \sup_y \sup_{\Delta_1} \sum_{i=0}^{n-1} |f(x_i, y) - f(x_{i+1}, y)|^p \leq K, \\ V_2(f)_p &= \sup_x \sup_{\Delta_2} \sum_{j=0}^{m-1} |f(x, y_j) - f(x, y_{j+1})|^p \leq K. \end{aligned}$$

Given a function $f(x, y)$, periodic in both variables with period 1. Denote by

$$\begin{aligned} \Delta_{h_1} f(x, y)_1 &= f(x + h_1, y) - f(x, y), \\ \Delta_{h_2} f(x, y)_2 &= f(x, y + h_2) - f(x, y), \\ \Delta_{h_1, h_2} f(x, y) &= \Delta_{h_1}(\Delta_{h_2} f(x, y)_2)_1 = \Delta_{h_2}(\Delta_{h_1} f(x, y)_1)_2 \\ &= f(x, y) - f(x + h_1, y) - f(x, y + h_2) + f(x + h_1, y + h_2). \end{aligned}$$

2. MAIN RESULTS

The main results of this paper are presented in the following propositions.

Theorem 1. *Let $f \in \text{PBV}_p(I^2)$, $p \geq 1$ and $\beta > \frac{2p}{1+p}$. Then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |C_{n,m}(f)|^{\beta} < \infty.$$

Theorem 2. *Let $f \in \text{PBV}_p(I^2)$, $p \geq 1$ and $\alpha < \frac{1}{2p} - \frac{1}{2}$. Then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [(n+1)(m+1)]^{\alpha} |C_{n,m}(f)| < \infty.$$

Theorem 3. *Let $f \in \text{PBV}_p(I^2)$, $p \geq 1$ and $\beta > 0, \alpha + 1 < \beta \left(\frac{1}{2p} + \frac{1}{2} \right)$. Then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [(n+1)(m+1)]^{\alpha} |C_{n,m}(f)|^{\beta} < \infty.$$

Since $\text{Lip } \frac{1}{p} \subset \text{PBV}_p$ in case $p > 1$ the sharpness of Theorems 1-3 follows from the works [12, 13].

3. AUXILIARY RESULTS

Lemma 1. *Let $f \in \text{PBV}_p(I^2)$, $p \geq 1$. Then*

$$\omega_i(\delta, f)_p \leq 3^{\frac{1}{p}} \delta^{\frac{1}{p}} V_i(f)_p \quad (i = 1, 2), 0 < \delta < 1,$$

where $V_i(f)_p$ is a partial p -variation of function.

Using the method of [8], we can easily obtain the validity of Lemma 1.

4. PROOF OF MAIN RESULTS

Proof of Theorem 1. We write

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |C_{n,m}(f)|^{\beta} = \sum_{n=0}^{\infty} |C_{n,0}(f)|^{\beta} + \sum_{m=1}^{\infty} |C_{0,m}(f)|^{\beta} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |C_{n,m}(f)|^{\beta}.$$

Let $n = 2^{n_1} + i$, $m = 2^{m_1} + j$, $n_1 = 0, 1, \dots$, $i = 1, \dots, 2^{n_1}$, $m_1 = 0, 1, \dots$, $j = 1, \dots, 2^{m_1}$. Then using Hölder inequality, from the Lemma 1 we get

$$\begin{aligned}
 & \sum_{i=1}^{2^{n_1}} |C_{2^{n_1}+i,0}(f)|^p \\
 &= 2^{\frac{pn_1}{2}} \sum_{i=1}^{2^{n_1}} \left| \int_0^1 \left(\int_{\frac{2i-2}{2^{n_1}+1}}^{\frac{2i-1}{2^{n_1}+1}} \left[f(x, y) - f\left(x + \frac{1}{2^{n_1}+1}, y\right) \right] dx \right) dy \right|^p \\
 &\leq 2^{\frac{pn_1}{2}} \sum_{i=1}^{2^{n_1}} \left[\int_0^1 \left(\int_{\frac{2i-2}{2^{n_1}+1}}^{\frac{2i-1}{2^{n_1}+1}} \left| \Delta_{\frac{1}{2^{n_1}+1}} f(x, y)_1 \right| dx \right) dy \right]^p \\
 &\leq 2^{\frac{pn_1}{2}} \sum_{i=1}^{2^{n_1}} \left[\left(\int_0^1 \left(\int_{\frac{2i-2}{2^{n_1}+1}}^{\frac{2i-1}{2^{n_1}+1}} \left| \Delta_{\frac{1}{2^{n_1}+1}} f(x, y)_1 \right|^p dx \right) dy \right)^{\frac{1}{p}} \left(\int_0^1 \int_{\frac{2i-2}{2^{n_1}+1}}^{\frac{2i-1}{2^{n_1}+1}} 1 dx dy \right)^{1-\frac{1}{p}} \right]^p \\
 &\leq 2^{\frac{pn_1}{2}} \frac{1}{2^{n_1(p-1)}} \int_0^1 \int_0^1 \left| \Delta_{\frac{1}{2^{n_1}+1}} f(x, y)_1 \right|^p dx dy \\
 &\leq 2^{n_1(1-\frac{p}{2})} \omega_1^p \left(\frac{1}{2^{n_1+1}}, f \right)_p \leq 2^{n_1(1-\frac{p}{2})} 3 \frac{1}{2^{n_1}} V_1^p(f)_p \leq c 2^{-\frac{n_1 p}{2}} V_1^p(f)_p.
 \end{aligned}
 \tag{2}$$

Let $\frac{2p}{1+p} < \beta < p$. Using Hölder inequality, from (2) we get

$$\begin{aligned}
 & \sum_{i=1}^{2^{n_1}} |C_{n_1,0}^{(i)}(f)|^\beta \leq \left(\sum_{i=1}^{2^{n_1}} |C_{n_1,0}^{(i)}(f)|^p \right)^{\frac{\beta}{p}} 2^{n_1(1-\frac{\beta}{p})} \\
 &\leq 2^{n_1(1-\frac{\beta}{p})} \left(c 2^{-\frac{n_1 p}{2}} V_1^p(f)_p \right)^{\frac{\beta}{p}} \\
 &\leq c 2^{n_1(1-\frac{\beta}{p})} 2^{-\frac{n_1 \beta}{2}} \leq c 2^{n_1[1-\frac{\beta}{p}-\frac{\beta}{2}]}.
 \end{aligned}
 \tag{3}$$

By (3) and from the condition of the Theorem 1 we obtain

$$\sum_{n=2}^{\infty} |C_{n,0}(f)|^\beta = \sum_{n_1=0}^{\infty} \sum_{i=1}^{2^{n_1}} |C_{2^{n_1}+i,0}(f)|^\beta \leq \sum_{n_1=0}^{\infty} 2^{n_1[1-\frac{\beta}{p}-\frac{\beta}{2}]} < \infty.$$

Analogously, we obtain that

$$\sum_{m=1}^{\infty} |C_{0,m}(f)|^\beta < \infty, \text{ for } \beta > \frac{2p}{1+p}.$$

Using Hölder inequality, by (1) and from Lemma 1 we get

$$\begin{aligned}
 & \sum_{i=0}^{2^{n_1}-1} \sum_{j=0}^{2^{m_1}-1} \left| \int_{\frac{2i}{2^{n_1}}}^{\frac{i+1}{2^{n_1}}} \int_{\frac{2j}{2^{m_1}}}^{\frac{j+1}{2^{m_1}}} f(x, y) \chi_{2^{n_1}+i}(x) \chi_{2^{m_1}+j}(y) dx dy \right|^p \\
 & \leq 2^{p \frac{n_1+m_1}{2}} \sum_{i=0}^{2^{n_1}-1} \sum_{j=0}^{2^{m_1}-1} \left[\int_{\frac{2i}{2^{n_1}+1}}^{\frac{2i+1}{2^{n_1}+1}} \int_{\frac{2j}{2^{m_1}+1}}^{\frac{2j+1}{2^{m_1}+1}} \left| \Delta_{\frac{1}{2^{n_1}+1}, \frac{1}{2^{m_1}+1}} f(x, y) \right| dx dy \right]^p \\
 & \leq 2^{p \frac{n_1+m_1}{2}} \sum_{i=0}^{2^{n_1}-1} \sum_{j=0}^{2^{m_1}-1} \left[\left(\int_{\frac{2i}{2^{n_1}+1}}^{\frac{2i+1}{2^{n_1}+1}} \int_{\frac{2j}{2^{m_1}+1}}^{\frac{2j+1}{2^{m_1}+1}} \left| \Delta_{\frac{1}{2^{n_1}+1}, \frac{1}{2^{m_1}+1}} f(x, y) \right|^p dx dy \right)^{\frac{1}{p}} \right. \\
 & \quad \times \left. \left(\int_{\frac{2i}{2^{n_1}+1}}^{\frac{2i+1}{2^{n_1}+1}} \int_{\frac{2j}{2^{m_1}+1}}^{\frac{2j+1}{2^{m_1}+1}} 1 dx dy \right)^{1-\frac{1}{p}} \right]^p \\
 & \leq 2^{p \frac{n_1+m_1}{2}} \frac{1}{2^{(n_1+m_1)(p-1)}} \int_0^1 \int_0^1 \left| \Delta_{\frac{1}{2^{n_1}+1}, \frac{1}{2^{m_1}+1}} f(x, y) \right|^p dx dy \\
 & \leq 2^{(n_1+m_1)(1-\frac{p}{2})} \omega_{1,2}^p \left(\frac{1}{2^{n_1}+1}, \frac{1}{2^{m_1}+1}, f \right)_p \\
 & \leq 2^{(n_1+m_1)(1-\frac{p}{2})} 2^p \omega_1^{\frac{p}{2}} \left(\frac{1}{2^{n_1}+1}, f \right)_p \omega_2^{\frac{p}{2}} \left(\frac{1}{2^{m_1}+1}, f \right)_p \\
 & \leq c 2^{(n_1+m_1)(1-\frac{p}{2})} \frac{2^p}{2^{\frac{n_1+m_1}{2}}} \leq c 2^{(n_1+m_1)(\frac{1}{2}-\frac{p}{2})}.
 \end{aligned}
 \tag{4}$$

Let $\frac{2p}{1+p} < \beta < p$. Using Hölder inequality, by (4) we write

$$\begin{aligned}
 & \sum_{i=0}^{2^{n_1}-1} \sum_{j=0}^{2^{m_1}-1} |C_{2^{n_1}+i, 2^{m_1}+j}(f)|^\beta \leq c 2^{(n_1+m_1)(1-\frac{p}{2})} \frac{\beta}{p} 2^{(n_1+m_1)(1-\frac{\beta}{p})} \\
 & = c 2^{(n_1+m_1)[\frac{\beta}{2p}-\frac{\beta}{2}+1-\frac{\beta}{p}]} \\
 & = c 2^{n_1[1-\frac{\beta}{2}-\frac{\beta}{2p}]} 2^{m_1[1-\frac{\beta}{2}-\frac{\beta}{2p}]}.
 \end{aligned}
 \tag{5}$$

By (5) and from the condition of the Theorem 1 we get

$$\begin{aligned}
 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |C_{n,m}(f)|^\beta &= \sum_{n_1=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{i=0}^{2^{n_1}-1} \sum_{j=0}^{2^{m_1}-1} |C_{2^{n_1}+i, 2^{m_1}+j}(f)|^\beta \\
 &\leq c \sum_{n_1=0}^{\infty} 2^{n_1[1-\frac{\beta}{2}-\frac{\beta}{2p}]} \sum_{m_1=0}^{\infty} 2^{m_1[1-\frac{\beta}{2}-\frac{\beta}{2p}]} < \infty.
 \end{aligned}$$

The proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. We write

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [(n+1)(m+1)]^{\alpha} |C_{n,m}(f)| &= \sum_{n=0}^{\infty} (n+1)^{\alpha} |C_{n,0}(f)| \\ &+ \sum_{m=1}^{\infty} (m+1)^{\alpha} |C_{0,m}(f)| + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [(n+1)(m+1)]^{\alpha} |C_{n,m}(f)|. \end{aligned}$$

Let $\beta = 1$. Then from (3) we get

$$\begin{aligned} (6) \quad \sum_{i=1}^{2^{n_1}} (2^{n_1} + i + 1)^{\alpha} |C_{2^{n_1}+i,0}(f)| &\leq c 2^{n_1 \alpha} \sum_{i=1}^{2^{n_1}} |C_{2^{n_1}+i,0}(f)| \\ &\leq c 2^{n_1 \alpha} 2^{n_1(\frac{1}{2}-\frac{1}{p})} = c 2^{n_1(\alpha+\frac{1}{2}-\frac{1}{p})}. \end{aligned}$$

By (6) and from the condition of the Theorem 2 we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)^{\alpha} |C_{n,0}(f)| &= \sum_{n_1=0}^{\infty} \sum_{i=1}^{2^{n_1}} (2^{n_1} + i + 1)^{\alpha} |C_{2^{n_1}+i,0}(f)| \\ &\leq c \sum_{n_1=0}^{\infty} 2^{n_1 \alpha} \sum_{i=1}^{2^{n_1}} |C_{2^{n_1}+i,0}(f)| \leq c \sum_{n_1=0}^{\infty} 2^{n_1(\alpha+\frac{1}{2}-\frac{1}{p})} < \infty. \end{aligned}$$

Analogously, we obtain that

$$\sum_{m=1}^{\infty} (m+1)^{\alpha} |C_{0,m}(f)| < \infty, \text{ for } \alpha < \frac{1}{2p} - \frac{1}{2}.$$

Let $\beta = 1$. Then by (5) and from the condition of the Theorem 2 we get

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [(n+1)(m+1)]^{\alpha} |C_{n,m}(f)| \\ \leq \sum_{n_1=0}^{\infty} \sum_{m_1=0}^{\infty} 2^{(n_1+m_1)\alpha} \sum_{i=1}^{2^{n_1}} \sum_{j=1}^{2^{m_1}} |C_{2^{n_1}+i,2^{m_1}+j}(f)| \\ \leq c \sum_{n_1=0}^{\infty} 2^{n_1(\alpha+\frac{1}{2}-\frac{1}{2p})} \sum_{m_1=0}^{\infty} 2^{m_1(\alpha+\frac{1}{2}-\frac{1}{2p})} < \infty. \end{aligned}$$

The proof of Theorem 2 is complete. $\square$

Combining the methods of Theorems 1-2 we can prove validity of Theorem 3. Observe that the result of this paper can be proved in the same way for dimension more than 2.

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