

ON CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. In the present paper, the authors introduce a new subclass $\mathcal{K}_s(\alpha, \beta)$ of close-to-convex functions. Several coefficient inequalities, growth, distortion and covering theorem for this class are provided.

1. INTRODUCTION

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic and univalent in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the usual subclasses of $\mathcal{S}$ whose members are close-to-convex and starlike in $\mathcal{U}$, respectively. Also let $\mathcal{S}^*(\alpha)$ denote the class of starlike functions of order α , $0 \leq \alpha < 1$.

Sakaguchi [5] once introduced a class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points, it consists of functions $f(z) \in \mathcal{S}$ satisfying

$$\Re \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0 \quad (z \in \mathcal{U}).$$

In a later paper, Gao and Zhou [1] discussed a class $\mathcal{K}_s$ of analytic functions related to the starlike functions, that is the subclass of $f(z) \in \mathcal{S}$ satisfying the following inequality

$$\Re \left\{ \frac{z^2 f'(z)}{g(z)g(-z)} \right\} < 0 \quad (z \in \mathcal{U}),$$

where $g(z) \in \mathcal{S}^*(\frac{1}{2})$.

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Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$ (see [3]).

In the present paper, we introduce the following class of analytic functions, and obtain some interesting results.

Definition 1. Let $\mathcal{K}_s(\alpha, \beta)$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$(1.1) \quad \left| \frac{z^2 f'(z)}{g(z)g(-z)} + 1 \right| < \beta \left| \frac{\alpha z^2 f'(z)}{g(z)g(-z)} - 1 \right| \quad (z \in \mathcal{U}),$$

where $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$ and $g(z) \in \mathcal{S}^*(\frac{1}{2})$.

It is easy to know that $\mathcal{K}_s(1, 1) = \mathcal{K}_s$, so $\mathcal{K}_s(\alpha, \beta)$ is a generalization of $\mathcal{K}_s$.

In the present paper, we shall provide several coefficient inequalities, growth, distortion and covering theorem for the class $\mathcal{K}_s(\alpha, \beta)$.

2. COEFFICIENT ESTIMATE

First we give a meaningful conclusion about the class $\mathcal{K}_s(\alpha, \beta)$.

Theorem 1. A function $f(z) \in \mathcal{K}_s(\alpha, \beta)$ if and only if

$$(2.1) \quad -\frac{z^2 f'(z)}{g(z)g(-z)} \prec \frac{1 + \beta z}{1 - \alpha \beta z} \quad (z \in \mathcal{U}).$$

Proof. Let $f(z) \in \mathcal{K}_s(\alpha, \beta)$, then from (1.1) we have

$$\left| \frac{z^2 f'(z)}{-g(z)g(-z)} - 1 \right|^2 < \beta^2 \left| \frac{\alpha z^2 f'(z)}{-g(z)g(-z)} + 1 \right|^2.$$

Expanding it we get

$$(1 - \alpha^2 \beta^2) \left| \frac{z^2 f'(z)}{-g(z)g(-z)} \right|^2 - 2(1 + \alpha \beta^2) \Re \left\{ \frac{z^2 f'(z)}{-g(z)g(-z)} \right\} < \beta^2 - 1.$$

If $\alpha \neq 1$ or $\beta \neq 1$, we have

$$\begin{aligned} \left| \frac{z^2 f'(z)}{-g(z)g(-z)} \right|^2 - \frac{2(1 + \alpha \beta^2)}{1 - \alpha^2 \beta^2} \Re \left\{ \frac{z^2 f'(z)}{-g(z)g(-z)} \right\} + \left(\frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} \right)^2 \\ < \frac{\beta^2 - 1}{1 - \alpha^2 \beta^2} + \left(\frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} \right)^2, \end{aligned}$$

that is,

$$\left| \frac{z^2 f'(z)}{-g(z)g(-z)} - \frac{1 + \alpha \beta^2}{1 - \alpha^2 \beta^2} \right|^2 < \frac{\beta^2(1 + \alpha)^2}{(1 - \alpha^2 \beta^2)^2},$$

or equivalently,

$$\left| \frac{z^2 f'(z)}{-g(z)g(-z)} - \frac{1 + \alpha\beta^2}{1 - \alpha^2\beta^2} \right| < \frac{\beta(1 + \alpha)}{1 - \alpha^2\beta^2}.$$

This tells us that the value region of $G(z) = (z^2 f'(z))/(-g(z)g(-z))$ is contained in the disk whose center is $(1 + \alpha\beta^2)/(1 - \alpha^2\beta^2)$ and radius is $[\beta(1 + \alpha)]/(1 - \alpha^2\beta^2)$. And we know that the function $\omega = q(z) = (1 + \beta z)/(1 - \alpha\beta z)$ maps the unit disk to the disk:

$$\left| \omega - \frac{1 + \alpha\beta^2}{1 - \alpha^2\beta^2} \right| < \frac{\beta(1 + \alpha)}{1 - \alpha^2\beta^2}.$$

Notice that $G(0) = q(0)$, $G(\mathcal{U}) \subset q(\mathcal{U})$, and $q(z)$ is univalent in $\mathcal{U}$, we obtain the following conclusion

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec q(z) = \frac{1 + \beta z}{1 - \alpha\beta z}.$$

Conversely, let

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \frac{1 + \beta z}{1 - \alpha\beta z},$$

then

$$(2.2) \quad -\frac{z^2 f'(z)}{g(z)g(-z)} = \frac{1 + \beta\omega(z)}{1 - \alpha\beta\omega(z)},$$

where $\omega(z)$ is analytic in $\mathcal{U}$, and $\omega(0) = 0$, $|\omega(z)| < 1$. By calculation we can easily obtain from (2.2) that

$$\left| \frac{z^2 f'(z)}{g(z)g(-z)} + 1 \right| < \beta \left| \frac{\alpha z^2 f'(z)}{g(z)g(-z)} - 1 \right|,$$

that is $f(z) \in \mathcal{K}_s(\alpha, \beta)$.

If $\alpha = \beta = 1$, inequality (1.1) becomes

$$\left| \frac{z^2 f'(z)}{-g(z)g(-z)} - 1 \right| < \left| \frac{z^2 f'(z)}{-g(z)g(-z)} + 1 \right|.$$

It is obvious that

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \frac{1 + z}{1 - z}.$$

This completes the proof of Theorem 1. □

Remark 1. From Theorem 1 we know that

$$(2.3) \quad \Re \left\{ \frac{zf'(z)}{(-g(z)g(-z))/z} \right\} > 0 \quad (z \in \mathcal{U}),$$

because of

$$\Re \left\{ \frac{1 + \beta z}{1 - \alpha\beta z} \right\} > 0 \quad (z \in \mathcal{U}).$$

In order to give the coefficient estimate of functions belonging to the class $\mathcal{K}_s(\alpha, \beta)$, we shall require the following two lemmas.

Lemma 1 ([1]). *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(\frac{1}{2})$, then*

$$\frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^*,$$

where

$$(2.4) \quad B_{2n-1} = 2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + (-1)^n 2b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2 \quad (n = 2, 3, \dots).$$

Remark 2. From Lemma 1 and inequality (2.3), we know that if $f(z) \in \mathcal{K}_s(\alpha, \beta)$, then $f(z)$ is a close-to-convex function. So $\mathcal{K}_s(\alpha, \beta)$ is a subclass of the class of close-to-convex functions.

Lemma 2 ([6]). *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, and satisfy the inequality*

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < \beta \left| \frac{\alpha z f'(z)}{g(z)} + 1 \right| \quad (z \in \mathcal{U}),$$

where $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, then for $n \geq 2$, we have

$$(2.5) \quad |na_n - b_n|^2 \leq 2(1 + \alpha\beta^2) \sum_{k=1}^{n-1} k |a_k| |b_k| \quad (|a_1| = |b_1| = 1).$$

Now we give the following theorem.

Theorem 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$, and satisfy the inequality (1.1), then for $n \geq 2$, we have*

$$(2.6) \quad |na_n - B_{2n-1}|^2 \leq 2(1 + \alpha\beta^2) \sum_{k=1}^{n-1} k |a_k| |B_{2k-1}| \quad (|a_1| = |B_1| = 1),$$

where B_{2n-1} is given by (2.4).

Proof. It is easy to know that inequality (1.1) can be written as

$$\left| \frac{zf'(z)}{(-g(z)g(-z))/z} - 1 \right| < \beta \left| \frac{\alpha z f'(z)}{(-g(z)g(-z))/z} + 1 \right|. \quad (2.7)$$

By Lemma 1, we have

$$\frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \subset \mathcal{S}.$$

Now, suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ and satisfy (2.7), so $f(z)$ and $(-g(z)g(-z))/z$ satisfy the condition of Lemma 2, thus, from (2.5), we can get (2.6). $\square$

3. SUFFICIENT CONDITION

In this section, we give the sufficient condition for functions belonging to the class $\mathcal{K}_s(\alpha, \beta)$.

Theorem 3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in $\mathcal{U}$, if for $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, we have*

$$(3.1) \quad \sum_{n=2}^{\infty} n(1 + \alpha\beta) |a_n| + \sum_{n=2}^{\infty} (1 + \beta) |B_{2n-1}| \leq (1 + \alpha)\beta,$$

where B_{2n-1} is given by (2.4), then $f(z) \in \mathcal{K}_s(\alpha, \beta)$.

Proof. By Lemma 1, we have

$$\frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^*.$$

Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then for $z \in \mathcal{U}$, we have

$$\begin{aligned} M &= \left| z f'(z) - \frac{-g(z)g(-z)}{z} \right| - \beta \left| \alpha z f'(z) + \frac{-g(z)g(-z)}{z} \right| \\ &= \left| z + \sum_{n=2}^{\infty} n a_n z^n - z - \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \right| \\ &\quad - \beta \left| \alpha z + \sum_{n=2}^{\infty} n \alpha a_n z^n + z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \right|. \end{aligned}$$

Now, for $|z| = r < 1$, we have

$$\begin{aligned} M &\leq \sum_{n=2}^{\infty} n |a_n| r^n + \sum_{n=2}^{\infty} |B_{2n-1}| r^{2n-1} \\ &\quad - \beta \left[(1 + \alpha)r - \sum_{n=2}^{\infty} n \alpha |a_n| r^n - \sum_{n=2}^{\infty} |B_{2n-1}| r^{2n-1} \right] \\ &< \left[-(1 + \alpha)\beta + \sum_{n=2}^{\infty} n(1 + \alpha\beta) |a_n| + \sum_{n=2}^{\infty} (1 + \beta) |B_{2n-1}| \right] r. \end{aligned}$$

From (3.1) we know that $M < 0$, thus we have

$$\left| \frac{z^2 f'(z)}{g(z)g(-z)} + 1 \right| < \beta \left| \frac{\alpha z^2 f'(z)}{g(z)g(-z)} - 1 \right| \quad (z \in \mathcal{U}),$$

that is $f(z) \in \mathcal{K}_s(\alpha, \beta)$, and the proof is complete. $\square$

4. GROWTH, DISTORTION AND COVERING THEOREM

Finally, we provide the growth, distortion and covering theorem for the class $\mathcal{K}_s(\alpha, \beta)$. For the purpose of this section, assume that the function $\phi(z)$ is an analytic function with positive real part in the unit disk $\mathcal{U}$, $\phi(\mathcal{U})$ is convex and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$.

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathcal{U}),$$

which satisfy the condition $\Re\{p(z)\} > 0$. A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(\phi)$ if

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $\phi(z) \in \mathcal{P}$. The class $\mathcal{S}^*(\phi)$ and a corresponding convex class $\mathcal{K}(\phi)$ were defined by Ma and Minda [2]. And the results about the convex class $\mathcal{K}(\phi)$ can be easily obtained from the corresponding results of functions in $\mathcal{S}^*(\phi)$. The functions $k_{\phi n}(z)$ ($n = 2, 3, \dots$) defined by $k_{\phi n}(0) = k'_{\phi n}(0) - 1 = 0$ and

$$1 + \frac{zk''_{\phi n}(z)}{k'_{\phi n}(z)} = \phi(z^{n-1})$$

are important examples of functions in $\mathcal{K}(\phi)$. The functions $h_{\phi n}(z)$ satisfying $h_{\phi n}(z) = zk'_{\phi n}(z)$ are examples of functions in $\mathcal{S}^*(\phi)$. Write $k_{\phi 2}(z)$ simply as $k_{\phi}(z)$ and $h_{\phi 2}(z)$ simply as $h_{\phi}(z)$.

A function $f(z) \in \mathcal{A}$ is in the class $\mathcal{K}_s(\phi)$ if

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \phi(z) \quad (z \in \mathcal{U}),$$

where $g(z) \in \mathcal{S}^*(\frac{1}{2})$ and $\phi(z) \in \mathcal{P}$.

In order to prove our next theorem, we shall require the following lemma. The proof of Lemma 3 below is much akin to that of Theorem 7 in [4], here we omit the details.

Lemma 3. *Let $\min_{|z|=r} |\phi(z)| = \phi(-r)$, $\max_{|z|=r} |\phi(z)| = \phi(r)$, $|z| = r < 1$. If $f(z) \in \mathcal{K}_s(\phi) \subset \mathcal{K}$, then we have*

$$h'_{\phi}(-r) \leq |f'(z)| \leq h'_{\phi}(r), \quad -h_{\phi}(-r) \leq |f(z)| \leq h_{\phi}(r),$$

and

$$f(\mathcal{U}) \supset \{\omega : |\omega| \leq -h(-1)\}.$$

These results are sharp.

Theorem 4. *Let $\min_{|z|=r} \left| \frac{1+\beta z}{1-\alpha\beta z} \right| = \psi(-r)$, $\max_{|z|=r} \left| \frac{1+\beta z}{1-\alpha\beta z} \right| = \psi(r)$, $|z| = r < 1$. If $f(z) \in \mathcal{K}_s(\alpha, \beta)$, then we have*

$$h'_{\psi}(-r) \leq |f'(z)| \leq h'_{\psi}(r), \quad -h_{\psi}(-r) \leq |f(z)| \leq h_{\psi}(r),$$

and

$$f(\mathcal{U}) \supset \{\omega : |\omega| \leq -h(-1)\}.$$

These results are sharp.

Proof. Setting $\phi(z) = \frac{1+\beta z}{1-\alpha\beta z}$ in Lemma 3, we can get the assertion of Theorem 4 easily. $\square$

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