

SECOND ORDER PARALLEL TENSORS ON α – SASAKIAN MANIFOLD

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ABSTRACT. Levy had proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [12] has proved that a second order parallel tensor in a Kaehler space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kaehlarian metric and the fundamental 2 – form. In this paper we show that a second order symmetric parallel tensor on an $\alpha - K$ contact ($\alpha \in R_o$) manifold is a constant multiple of the associated metric tensor and we also prove that there is no nonzero skew symmetric second order parallel tensor on an $\alpha -$ Sasakian manifold.

1. INTRODUCTION

In 1923, Eisenhart [10] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of the metric tensor is reducible. In 1926, Levy [11] had obtained the necessary and sufficient conditions for the existence of such tensors, Recently Sharma [12] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non singular) tensor on an $n -$ dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [12] that on a Sasakian manifold there is no nonzero parallel 2 – form. In this paper we have considered an almost contact metric manifold and have proved the following two theorems.

Theorem 1.1. *On an $\alpha - K$ contact ($\alpha \in R_o$) manifold a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

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Now the question arises whether there is a skew symmetric second order parallel tensor on a $\alpha - k$ contact manifold. We do not have an answer to it. However we do have an answer if the manifold is $\alpha -$ Sasakian where $\alpha \in R_0$.

Theorem 1.2. *On an $\alpha -$ Sasakian manifold there is no nonzero parallel 2 - forms.*

2. PRELIMINARIES

A C^∞ manifold M of dimension $2n + 1$ is called a contact manifold if it carries a global 1 - form A such that $A \wedge (dA)^n \neq 0$. On a contact manifold there exists a unique vector field T called the characteristic vector field such that

$$(2.1) \quad A(T) = 1, \quad (dA)(T, X) = 0$$

for any vector field X on M . By polarization we obtain a Riemannian metric g called an associated metric and a $(1, 1)$ tensor field ϕ on M such that

$$(2.2) \quad \begin{aligned} \phi^2 &= -I + A \otimes T \\ (dA)(X, Y) &= g(X, \phi Y) \\ A(X) &= g(X, T) \end{aligned}$$

for the arbitrary vector fields X and Y on M . If in addition to (2.1) and (2.2), M^n admits a positive definite Riemannian metric g such that

$$(2.3) \quad \begin{aligned} g(\phi X, \phi Y) &= g(X, Y) - A(X)A(Y) \\ \phi(T) &= 0, \quad A(\phi(X)) = 0, \forall X, Y \in \mathfrak{X}(M) \\ \text{and rank}(\phi) &= 2n \text{ everywhere on } M. \end{aligned}$$

Such a manifold satisfying (2.1), (2.2), and (2.3) is called an almost contact metric manifold. The structure endowed in M is called (ϕ, A, T, g) - structure.

For a (ϕ, A, T, g) - structure, the skew symmetric bilinear form

$$(2.4) \quad \Phi(X, Y) = g(X, \phi Y)$$

is called the fundamental 2 - form of the almost contact metric structure.

3. SOME DEFINITIONS AND THEOREMS

Definition 3.1. An almost contact metric structure is said to be an $\alpha - K$ contact structure if the vector field T is killing with respect to g .

In proving Theorems 1.1 and 1.2, we need the following theorems.

Theorem 3.1. *On an $\alpha - K$ contact structure the following holds.*

$$(3.1) \quad \nabla_X T = -\alpha \phi X \text{ for all } X \in \mathfrak{X}(M)$$

where ∇ is the Riemannian connection of g .

Theorem 3.2. *An almost contact metric structure (ϕ, A, T, g) is α - Sasakian iff*

$$(3.2) \quad (\nabla_x \phi) Y = \alpha \{g(X, Y) T - A(Y) X\}$$

where ∇ denotes the Riemannian connection of g .

Proof. The proofs of the above theorems follows in a similar fashion as in the Theorem 6.3 by Blair [3]. $\square$

Definition 3.2 ([2]). An almost α - Sasakian manifold M is an almost contact metric manifold such that $\phi(X, Y) = \frac{1}{\alpha} d\eta(X, Y)$, $\alpha \in R_0$ and M is a α - Sasakian manifold if the structure is normal.

Theorem 3.3. *An almost contact metric manifold M is α - Sasakian manifold iff for all $X, Y \in \mathfrak{X}(M)$*

$$(3.3) \quad R(X, Y) T = \alpha \{A(Y) X - A(X) Y\}$$

Proof. The proof of the above theorem follows in view of Lemma 6.1 of Blair [3]

The two conditions of being normal and contact metric may be written as the following:

$$(3.4) \quad R(T, X) Y = \alpha \{g(X, Y) T - A(Y) X\}$$

$\square$

Theorem 3.4. *For an α - K contact manifold we have*

$$(3.5) \quad R(T, X) T = \alpha \{-X + A(X) T\}$$

Proof. In view of (3.4), the proof follows immediately. $\square$

For a detailed study on a contact manifold the reader is referred to [2].

4. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. Let h denote a $(0, 2)$ - tensor field on an α - K contact manifold M such that $\nabla h = 0$. Then it follows that

$$(4.1) \quad h(R(W, X) Y, Z) + h(Y, R(W, X) Z) = 0$$

for arbitrary vector fields X, Y, Z, W on M .

We write (4.1) as follows

$$g(R(W, X) Y, Z) + g(Y, R(W, X) Z) = 0.$$

On substituting $W = Y = Z = T$ in (4.1) we get:

$$(4.2) \quad g(R(T, X) T, T) + g(T, R(T, X) T) = 0.$$

In view of Theorem (3.4), the above equation becomes:

$$(4.3) \quad g(-\alpha X + \alpha A(X) T, T) + g(T, -\alpha X + \alpha A(X) T) = 0.$$

In this equation, using (2.2) we get

$$(4.4) \quad 2\alpha g(X, T) h(T, T) - \alpha h(X, T) - \alpha h(T, X) = 0.$$

Differentiating (4.4) covariantly with respect to Y and using Theorem (3.1) we get

$$(4.5) \quad \begin{aligned} & 2\alpha h(T, T) g(\nabla_Y X, T) - 2\alpha^2 h(T, T) g(X, \phi Y) \\ & - \alpha g(\nabla_Y X, T) + \alpha^2 g(X, \phi Y) + \alpha^2 g(\phi Y, X) - \alpha g(T, \nabla_Y X) = 0. \end{aligned}$$

Replacing Y by ϕY and using equations (2.2), (2.3) and (4.4) we obtain

$$h(X, Y) + h(Y, X) = 2h(T, T) g(X, Y).$$

But h is symmetric, thus on simplifying the above equation we get

$$(4.6) \quad 2h(T, T) g(X, Y) = 2h(X, Y).$$

In view of the fact that $h(T, T)$ is constant by differentiating it along any vector on M^{2n+1} we get

$$h(T, T) g(X, Y) = h(X, Y)$$

which completes the proof. $\square$

Proof of Theorem 1.2. Let us consider h to be a parallel 2-form on an α -Sasakian manifold M^{2n+1} and let H be a $(1, 1)$ tensor field metrically equivalent to h since $h(X, Y) = g(HX, Y)$. Now (4.1) can be written as

$$(4.7) \quad g(R(W, X)Y, Z) + g(Y, R(W, X)Z) = 0.$$

Let us put $X = Y = T$ in (4.7) and using the fact that $h(X, Y) = g(HX, Y)$ we get

$$(4.8) \quad g(HR(W, T)T, Z) + g(HT, R(WT)Z) = 0.$$

Applying the skew symmetric property of $R(X, Y)$ and using (3.3) and (3.4) in (4.8) and after simplifying, we obtain

$$(4.9) \quad \alpha g(HZ, T)T + \alpha g(Z, T)HT = \alpha HZ.$$

Differentiating (4.9) along ϕX we obtain

$$(4.10) \quad \begin{aligned} & 2\alpha A(X)A(HZ)T - \alpha g(HZ, X)T - \alpha g(HZ, T)X \\ & = \alpha g(Z, X)HT - 2\alpha A(X)A(Z)HT + \alpha A(Z)HX. \end{aligned}$$

Let $\{e_i\}, i = 1, 2, \dots, 2n+1$ be an orthonormal basis of the tangent space. In the above equation (4.10), we substitute $X = e_i$ and take the inner product with e_i and eventually summing over i gives us

$$\alpha(2n-1)g(HZ, T) = 0.$$

Since $\alpha(2n-1) \neq 0$, we have $g(HZ, T) = 0$. But $g(HZ, T) = -g(HT, Z)$. Thus, $HT = 0$ and hence (4.9) shows that $H = 0$, which completes the proof. $\square$

REFERENCES

- [1] D. Blair and S. Goldberg. Topology of almost contact manifolds. *J. Differ. Geom.*, 1:347–354, 1967.
- [2] D. E. Blair. *Contact manifolds in Riemannian geometry.*, volume 509 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York., 1976.
- [3] D. E. Blair. *Riemannian geometry of contact and symplectic manifolds.*, volume 203 of *Progress in Mathematics*. Birkhäuser, Boston, MA., 2002.
- [4] L. Das. Fiberings on almost r -contact manifolds. *Publ. Math.*, 43:1–7, 1993.
- [5] L. Das. Prolongation of F -structure to the tangent bundle of order 2. *Int. J. Math. Math. Sci.*, 16(1):201–204, 1993.
- [6] L. Das. On CR-structures and F -structure satisfying $F^K + (-)^{K+1} F = 0$. *Rocky Mt. J. Math.*, 36, 2006.
- [7] L. Das and R. Nivas. On a differentiable manifold with $[F_1, F_2](K+1, 1)$ -structure. *Tensor (N.S.)*, 65:29–35, 2004.
- [8] L. Das and R. Nivas. On certain structures defined on the tangent bundle. *Rocky Mt. J. Math.*, 36, 2006.
- [9] L. S. Das. Invariant submanifolds of the manifold with $\phi(k, -(-)^{k+1})$ -structure. *Tensor (N.S.)*, 64(2):189–196, 2003.
- [10] L. Eisenhart. Symmetric tensors of the second order whose first covariant derivatives are zero. *Trans. Am. Math. Soc.*, 25:297–306, 1923.
- [11] H. Levy. Symmetric tensors of the second order whose covariant derivatives vanish. *Ann. Math.*, 27:91–98, 1926.
- [12] R. Sharma. Second order parallel tensor in real and complex space forms. *Int. J. Math. Math. Sci.*, 12:787–790, 1989.

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