Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
23(2) (2007), 161–165
www.emis.de/journals
ISSN 1786-0091

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF k-CHORDAL POLYGONS

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Abstract. The aim of this article is to extend the results that presented in [1, 2] for k-chordal polygons. Moreover, a conjecture that was stated in [1, 2] is disproved here by using the obtained results.

1. Introduction

k-chordal polygons are defined properly in [1, 2], where results concerning their existence are presented. Throughout the present article we use the same definitions and nomenclature as in [1, 2].

Let $\alpha_1, \ldots, \alpha_n$ be positive reals. A k-chordal polygon with sides $\alpha_1, \ldots, \alpha_n$ is denoted (see [1, 2]) $\underline{A} = A_1A_2\ldots A_n$ and the following angles are defined: $\beta_i = \angle CA_iA_{i+1}$ and the central angles $\theta_i = \angle A_iCA_{i+1}$ for $i = 1, \ldots, n$ (C is the center of the circum-circle). For a k-chordal polygon it holds that $\sum_{i=1}^{n} \theta_i = 2k\pi$, which means that the total arc of the polygon is k times the circumference of the circle.

The article is divided as follows: In Section 2 we present necessary and sufficient conditions for the existence of k-chordal polygons. In Section 3 we disprove a conjecture (hypothesis) that was stated in [1, 2] concerning a sufficient condition for the existence of k-chordal polygons.

2. Existence results

For the rest of the article we consider k-chordal polygons with sides $\alpha_1, \ldots, \alpha_n$ which are positive reals and without loss of generality we let $\alpha_1 = \max_{1\le i\le n} \alpha_i$. The following Corollary which gives a necessary condition is presented in [1, 2]:

Corollary 1. *If $\alpha_1, \ldots, \alpha_n$ are the sides of a k-chordal polygon then:*

$$\sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} > 2k - 1. \tag{1}$$

2000 *Mathematics Subject Classification.* 51E12.
Key words and phrases. k-chordal polygons, existence.

It is proved in [2] with the use of a counterexample that (1) is not a sufficient condition. Thus, the next Hypothesis is stated as a potential sufficient condition:

Hypothesis. Let the lengths $\alpha_1, \dots, \alpha_n$ be such that:

$$\sum_{i=2}^{n} \left(\frac{\alpha_i}{\alpha_1}\right)^{2m-1} > 2m - 1, \tag{2}$$

where $m = [\frac{n-1}{2}]$, i.e. $m = \frac{n-1}{2}$ if n is odd and $m = \frac{n}{2} - 1$ if n is even. Then for each $k = 1, \dots, m$ there exists a k-chordal polygon with side lengths $\alpha_1, \dots, \alpha_n$.

The authors in [2] note that it is difficult to prove the Hypothesis. Since they do not provide a proof, they put some additional assumptions in order to prove another existence theorem (Theorem 2.1 [2]). We will see in Section 3 that the Hypothesis is false.

Let us first present the following Lemma, which gives a sufficient condition for the existence of a k-chordal polygon.

Lemma 1. *Suppose that:*

$$\sum_{i=2}^{n} \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) > (2m-1)\frac{\pi}{2}. \tag{3}$$

Then for each $k = 1, \dots, m$ there exists a k-chordal polygon with sides $\alpha_1, \dots, \alpha_n$.

Proof. We use similar arguments to those in Theorem 2.1 [2]. We want to prove that for each $k = 1, \dots, m$ there are angles $\beta_1, \dots, \beta_n$ $(0 < \beta_i < \pi/2)$ such that:

$$\frac{\cos\beta_1}{\alpha_1} = \cdots = \frac{\cos\beta_n}{\alpha_n},$$
$$\sum_{i=1}^{n} \beta_i = (n-2k)\frac{\pi}{2}.$$

Let us first define certain angles $\gamma_i, i = 1, \dots, n$ such that: $\cos\gamma_i = \frac{\alpha_i}{\alpha_1}\cos\gamma_1$. Thus, $\gamma_i = \arccos\left(\frac{\alpha_i}{\alpha_1}\cos\gamma_1\right)$. Our aim is for each $k = 1, \dots, m$ to find a γ_1 $(0 < \gamma_1 < \pi/2)$ such that $\sum_{i=1}^{n} \gamma_i = (n-2k)\frac{\pi}{2}$.

For each $k = 1, \dots, m$, we define the following functions in one variable:

$$h_k(\gamma_1) = \gamma_1 + \sum_{i=2}^{n} \arccos\left(\frac{\alpha_i}{\alpha_1}\cos\gamma_1\right) - (n-2k)\frac{\pi}{2}.$$

It is now enough to show that for each $k = 1, \dots, m$ there is a $\gamma_1 \in (0, \pi/2)$ such that $h_k(\gamma_1) = 0$. Since $h_k(\gamma_1)$ are continuous functions in $[0, \pi/2]$ with respect to γ_1, we simply need to prove that for each $k = 1, \dots, m$ we have $h_k(\pi/2) > 0$ and $h_k(0) < 0$.

First, we have

$$h_k(\frac{\pi}{2}) = \frac{\pi}{2} + (n-1)\frac{\pi}{2} - (n-2k)\frac{\pi}{2} = k\pi > 0.$$

Secondly, we have

$$\begin{aligned} h_k(0) &= \sum_{i=2}^{n} \arccos\left(\frac{\alpha_i}{\alpha_1}\right) - (n-2k)\frac{\pi}{2} \\ &= (n-1)\frac{\pi}{2} - \sum_{i=2}^{n} \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) - (n-2k)\frac{\pi}{2} \\ &= (2k-1)\frac{\pi}{2} - \sum_{i=2}^{n} \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) < 0. \end{aligned}$$

Here we have used the inequality (3). This completes the proof. □

Lemma 1 provides a sufficient condition for the existence of a k-chordal polygon. Note also that from inequality (3) we can get directly another sufficient condition. Thus, the next Corollary follows easily from Lemma 1.

Corollary 2. *Suppose that:*

$$\sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} \geq (2m-1)\frac{\pi}{2}. \tag{4}$$

Then for each $k = 1, \ldots, m$ there exists a k-chordal polygon with sides $\alpha_1, \ldots, \alpha_n$.

Proof. Simply we use the fact that $\arcsin(x) > x$ for $0 < x \leq 1$ and so inequality (4) implies inequality (3). □

It is interesting to compare the necessary condition (1) to the sufficient condition (4). It is clear that the right hand side of inequality (1) must be multiplied by a factor $\frac{\pi}{2} > 1$ in order to get the sufficient condition (4). Observe also that inequality (4) is only a sufficient condition for the existence of a k-chordal polygon and not a necessary one.

The question that arises naturally is if inequality (3) is also a necessary condition. In fact it is, and the next Theorem summarizes the results and provides a necessary and sufficient condition for the existence of k-chordal polygons.

Theorem 1. *For each $k = 1, \ldots, m$ there exists a k-chordal polygon with sides $\alpha_1, \ldots, \alpha_n$ if and only if*

$$\sum_{i=2}^{n} \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) > (2m-1)\frac{\pi}{2}.$$

Proof. Assume that a k-chordal polygon with sides $\alpha_1, \dots, \alpha_n$ exists. Then for the central angles θ_i we know that $\sum_{i=1}^n \theta_i = 2k\pi$ and since $\theta_i = 2\arcsin\left(\frac{\alpha_i}{2R_k}\right)$ we get

$$\sum_{i=1}^{n} \arcsin\left(\frac{\alpha_i}{2R_k}\right) = k\pi.$$

Where R_k is the circum-radius of the corresponding circum-circle. We also have that

$$\alpha_1 < 2R_k \iff \frac{\alpha_i}{\alpha_1} > \frac{\alpha_i}{2R_k} \iff \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) > \arcsin\left(\frac{\alpha_i}{2R_k}\right).$$

Thus,

$$\sum_{i=1}^{n} \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) > k\pi \iff \sum_{i=2}^{n} \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) > (2k-1)\frac{\pi}{2},$$

which holds for each $k = 1, \dots, m$. This proves the necessary part. For the sufficient part we use Lemma 1 and the proof is complete. $\square$

Before we disprove the Hypothesis in Section 3, let us note that Corollary 1 from [1, 2] could be proved directly from Theorem 1. Since $\frac{\pi}{2}x \geq \arcsin(x)$ for $0 \leq x \leq 1$, from the necessary part of Theorem 1 we get

$$\sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} \geq \frac{2}{\pi} \sum_{i=2}^{n} \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) > 2m - 1,$$

which is exactly inequality (1) from Corollary 1.

3. Disprove of the Hypothesis

In this Section we disprove the Hypothesis that posed in [1, 2] by using a counterexample and Theorem 1.

Let $\alpha_1 = 100$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 91$. We have a pentagon ($n = 5$) and we choose $m = \frac{n-1}{2} = 2$ as in [1, 2]. By direct calculation we get

$$\sum_{i=2}^{5} \left(\frac{\alpha_i}{\alpha_1}\right)^3 \simeq 3.014284 > 2m - 1 = 3.$$

Since inequality (2) holds, from the Hypothesis we must conclude that the 2-chordal pentagon with the given sides exists. On the other hand, by using Theorem 1 we have

$$\sum_{i=2}^{5} \arcsin\left(\frac{\alpha_i}{\alpha_1}\right) \simeq 4.5731362 < (2m-1)\frac{\pi}{2} \simeq 4.712389.$$

Thus, the 2-chordal pentagon with the given sides does not exist. This means that the Hypothesis is false and inequality (2) is not a sufficient condition for the existence of a k-chordal polygon.

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Received February 15, 2007.

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