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A SPECIAL NONLINEAR CONNECTION IN SECOND ORDER GEOMETRY

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Abstract. We show that, for mechanical system with external forces, the equations of deviations of solution curves of the corresponding Lagrange equations, determine a nonlinear connection on the second order tangent bundle. In particular, Jacobi equations in Finsler and Riemann spaces determine such a nonlinear connection.

1. Introduction

As shown in [27], nonlinear connections on bundles can be a powerful tool in integrating systems of differential equations. A way of obtaining them is that of deriving them from the respective systems of DE's, in particular, from variational principles, [2], [16], [15]. For instance, an ODE system of order 2 on a manifold M induces a nonlinear connection on its tangent bundle TM. A remarkable example is here the Cartan nonlinear connection of a Finsler space, which has the property that its autoparallel curves correspond to geodesics of the base manifold:

$$\frac{\delta y^i}{dt} := \frac{dy^i}{dt} + N^i{}_j y^j = 0.$$

Further, an ODE system of order three determines a nonlinear connection on the second order tangent (jet) bundle $T^2M = J_0^2(\mathbb{R}, M)$. For instance, Craig-Synge equations (R. Miron, [16])

$$\frac{d^3x^i}{dt^3} + 3!G^i(x, \dot{x}, \ddot{x}) = 0,$$

lead to:

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a) Miron's connection:

$$M_{(1)}{}^i_j = \frac{\partial G^i}{\partial y^{(2)j}}, \; M_{(2)}{}^i_j = \frac{1}{2}\left(S M_{(1)}{}^i_j + M_{(1)}{}^i_m M_{(1)}{}^m_j \right), \tag{1}$$

where $S = y^i \frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^i} - 3G^i \frac{\partial}{\partial y^{(2)i}}$ is a semispray on T^2M.

b) Bucătaru's connection

$$M_{(1)}{}^i{}_j = \frac{\partial G^i}{\partial y^{(2)j}}, M_{(2)}{}^i{}_j = \frac{\partial G^i}{\partial y^j}.$$

With respect to the last one, if G^i are the coefficients of a spray on T^2M (i.e., 3-homogeneous functions), then the Craig-Synge equations can be interpreted as:

$$\frac{\delta y^{(2)i}}{dt} = 0, \tag{2}$$

where $\frac{\delta y^{(2)i}}{dt} := \frac{dy^{(2)i}}{dt} + M_{(1)}{}^i{}_j \frac{dy^j}{dt} + M_{(2)}{}^i{}_j \frac{dx^j}{dt}$.

In Miron's and Bucătaru's approaches, nonlinear connections on T^2M are obtained from a Lagrangian of order 2, $L(x, \dot{x}, \ddot{x})$, by computing the first variation of its integral of action.

Here, we propose a different approach, which, we consider, could be at least as interesting as the above one from the point of view of Mechanics - namely, we start with a first order Lagrangian $L(x, \dot{x})$ and compute its second variation.

This way, for a mechanical system $(M, L(x, \dot{x}), F(x, \dot{x}))$ with external force field F, we obtain a nonlinear connection on T^2M, with respect to which the equations of deviations of evolution curves have a simple invariant form.

As a remark, our nonlinear connection is also suitable for modelling the solutions of a (globally defined) ODE system, not necessarily attached to a certain Lagrangian, together with the deviations of these solutions.

More precisely, in the following our aims are:

(1) to obtain the Jacobi equations for the trajectories

$$\frac{\delta y^i}{dt} = \frac{1}{2}F^i(x, y)$$

(for extremal curves of a 2-homogeneous Lagrangian $L(x, \dot{x})$ in presence of external forces).

(2) to build a nonlinear connection such that:

$$w \in \mathcal{X}(M) \text{ Jacobi field along } c \Leftrightarrow \frac{\delta w^{(2)i}}{dt} = 0,$$

where $\frac{d}{dt}$ denotes directional derivative with respect to $\dot{c}$ and

$$\frac{\delta w^{(2)i}}{dt} = \frac{1}{2}\frac{d^2 w^i}{dt^2} + \underset{(1)}{M^i}_j \frac{dw^j}{dt} + \underset{(2)}{M^i}_j w^j.$$

For $F = 0$, this nonlinear connection has as additional properties:

I. In Finsler spaces M, c is a geodesic of M if and only if its extension T^2M is horizontal.

II. A vector field w along a geodesic c on M is parallel along c if and only if $\frac{\delta w^i}{dt} = 0$.

Throughout the paper, by 'differentiable' or 'smooth' we mean $\mathcal{C}^\infty$-differentiable.

2. Tangent bundle of first and second order

Let M be a real differentiable manifold of dimension n and class $\mathcal{C}^\infty$; the coordinates of a point $x \in M$ in a local chart (U, ϕ) will be denoted by $\phi(x) = \left(x^i\right)$, $i = 1, \dots, n$. Let (TM, π, M) be its tangent bundle and (x^i, y^i) the coordinates of a point in a local chart.

The *2-tangent bundle* (T^2M, π^2, M) is the space of jets of order two at 0 of all smooth functions $f\colon (-\varepsilon, \varepsilon) \to M$, $t \mapsto (f^i(t))$, on $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$, ([19]-[24], [16], [10]).

In a local chart, a point p of T^2M will have the coordinates $(x^i, y^i, y^{(2)i})$. This is,

$$x^i = f^i(0), \quad y^i = \dot{f}^i(0), \quad y^{(2)i} = \frac{1}{2}\ddot{f}^i(0), \quad i = 1, \dots, n,$$

for some f as above. Then, $\left(T^2M, \pi^2, M\right)$ is a differentiable manifold of class $\mathcal{C}^\infty$ and dimension $3n$, and TM can be identified with a submanifold of T^2M. The local coordinate changes induced by local coordinate changes on M are, [16], [19]-[24],

$$\begin{aligned}
\widetilde{x}^i &= \widetilde{x}^i\left(x^1, \dots, x^n\right), \ \det\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) \neq 0 \\
\widetilde{y}^i &= \frac{\partial \widetilde{x}^i}{\partial x^j} y^j \\
2\widetilde{y}^{(2)i} &= \frac{\partial \widetilde{y}^i}{\partial x^j} y^j + 2\frac{\partial \widetilde{y}^i}{\partial y^j} y^{(2)j}.
\end{aligned} \tag{3}$$

For a curve $c\colon [0,1] \to M$, $t \mapsto (x^i(t))$ on the base manifold M, let us denote:

- by $\widehat{c}$ its *extension* to the tangent bundle TM :

$$\widehat{c}\colon [0,1] \to M, t \mapsto (x^i(t), \dot{x}^i(t));$$

along $\widehat{c}$, there holds:

$$y^i = \dot{x}^i(t), \quad i = 1, \dots, n;$$

- by $\widetilde{c}$ its *extension to* T^2M:

$$\widetilde{c}\colon [0,1] \to T^2M, \qquad t \mapsto (x^i(t), \dot{x}^i(t), \frac{1}{2}\ddot{x}^i(t));$$

along such an extension curve, there holds

$$y^i(t) = \dot{x}^i(t), \quad y^{(2)i}(t) = \frac{1}{2}\ddot{x}^i(t), \quad i = 1, \dots, n.$$

A tensor field on TM (or T^2M) is called a *distinguished tensor field*, or simply, a *d-tensor field* if, under a change of local coordinates induced by a change of coordinates on the base manifold M, its components transform by the same rule as the components of a corresponding tensor field on M, [16].

3. Nonlinear connections on TM

Let (TM, π, M) be the tangent bundle of a differentiable manifold M as above and (x^i, y^i) the coordinates of a point $p \in TM$ in a local chart. For simplicity, we shall also denote $(x, y) = (x^i, y^i)_{i=\overline{1,n}}$.

Let $d\pi\colon T\,(TM) \to TM$ denote the tangent linear mapping of the projection $\pi\colon TM \to M$ and $V\,(TM) = \ker d\pi$, the *vertical subbundle* of $T\,(TM)$. Its fibres generate the *vertical distribution* V on TM of local dimension n, $V\colon p \in TM \mapsto V\,(p) \subset T_p\,(TM)$, locally spanned by $\{\frac{\partial}{\partial y^i}\}$.

A *nonlinear (Ehresmann) connection on* TM, [16], [18], is a distribution $N\colon p \in TM \mapsto N(p) \subset T_p(TM)$, which is supplementary to the vertical distribution:

$$T_p(TM) = N\,(p) \oplus V\,(p)\,, \quad \forall p \in TM. \tag{4}$$

Let

$$B = \left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\},$$

where:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j}, \quad i = 1, \dots, n, \tag{5}$$

denote a local adapted basis to the direct decomposition (4). The quantities $N^i{}_j = N^i{}_j(x, y)$, [16], [18], are called the *coefficients* of the nonlinear connection N.

With respect to local coordinate changes on TM induced by changes of local coordinates $(x^i) \mapsto (\tilde{x}^i)$ on the base manifold M, $\frac{\delta}{\delta x^i}$ transform by the rule: $\frac{\delta}{\delta x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\delta}{\delta \widetilde{x}^j}$.

The dual basis of B is $B^* = \{dx^i, \delta y^i\}$, given by

$$\delta y^i = dy^i + N^i{}_j dx^j. \tag{6}$$

With respect to changes of local coordinates on TM induced by local coordinate changes on M, there holds: $\delta \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}\delta y^j$.

Any vector field $X \in \mathcal{X}(TM)$ is represented in the local adapted basis as

$$X = X^{(0)i}\frac{\delta}{\delta x^i} + X^{(1)i}\frac{\partial}{\partial y^i}, \tag{7}$$

where the components $X^{(0)i}\dfrac{\delta}{\delta x^i}$ and $X^{(1)i}\dfrac{\partial}{\partial y^i}$ are d-vector fields.

Similarly, a 1-form $\omega \in \mathcal{X}^*(TM)$ will be decomposed as the sum of two d-1-forms:

$$\omega = \omega_i^{(0)} dx^i + \omega_i^{(1)}\delta y^i. \tag{8}$$

In particular, if $\widehat{c}$: $t \to (x^i(t), y^i(t))$ is an extension curve to TM, then its tangent vector field is expressed in the adapted basis as

$$\dot{\widehat{c}} = \frac{dx^i}{dt}\frac{\delta}{\delta x^i} + \frac{\delta y^i}{dt}\frac{\partial}{\partial y^i}. \tag{9}$$

In our further considerations, an important role will be played by the notions of *semispray* and *spray*, [25], [10]. A semispray $S \in \mathcal{X}(TM)$ is a vector field locally described in the natural basis by $S = y^i\dfrac{\partial}{\partial x^i} - 2G^i(x,y)\dfrac{\partial}{\partial y^i}$, where the functions G^i (called the *coefficients* of the semispray) obey, with respect to coordinate changes induced by a change of local coordinates $(x^i) \mapsto (\tilde{x}^i)$ on M, the rule: $2\tilde{G}^i = 2\dfrac{\partial \tilde{x}^i}{\partial x^j}G^j - \dfrac{\partial \tilde{y}^i}{\partial x^j}y^j$, $i = 1, \dots, n$. If G^i are 2-homogeneous functions in y, then the semispray is called a spray.

As shown by Grifone, [12], a semispray (in particular, a spray) on M determines a nonlinear connection on TM.

Also, evolution curves of mechanical systems with external forces, can be described in terms of semisprays on TM, (R. Miron, [15]):

Proposition 1. *Let $L = L(x, \dot{x})$ be a nondegenerate Lagrangian:*

$$\det\left(\frac{\partial^2 L}{\partial y^i \partial y^j}\right) \neq 0,$$

and $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i \partial y^j}$, the induced (Lagrange) metric tensor. Then, the equations of evolution of a mechanical system with the Lagrangian L and the external force field $F = F_i(x, \dot{x})dx^i$ are

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = \frac{1}{2}F^i(x, \dot{x}), \tag{10}$$

where

$$2G^i = \frac{1}{2} g^{is} \left(\frac{\partial^2 L}{\partial y^s \partial x^j} y^j - \frac{\partial L}{\partial x^s} \right),$$

yield a semispray (called the canonical semispray of the Lagrange space (M, L)*) and* $F^i = g^{ij} F_j$, $i = 1, \ldots, n$.

In the following, we shall use the above results in the case when G is a *spray*; this is, we shall have

$$2G^i = \frac{\partial G^i}{\partial y^j} y^j.$$

Then, [12], [2], [5], [18], the quantities

$$N^i_{\ j} = \frac{\partial G^i}{\partial y^j}$$

are the coefficients of a nonlinear connection on TM. Moreover, $N^i_{\ j} = N^i_{\ j}(x, y)$ are 1-homogeneous in y.

With respect to the above nonlinear connection, equations (10) take the form:

$$\frac{\delta y^i}{dt} = \frac{1}{2} F^i, \quad i = 1, \ldots, n. \tag{11}$$

In particular, if there are no external forces, this is, if $F^i = 0$, then the extremal curves $t \mapsto x^i(t)$ of the Lagrangian L have horizontal extensions and vice-versa: horizontal extension curves $\widehat{c}$ project onto solution curves of the Euler-Lagrange equations of L.

4. Nonlinear connections on T^2M

Let $d\pi^2 \colon T\left(T^2 M\right) \to TM$ denote the tangent linear mapping of the projection $\pi^2 \colon T^2 M \to M$ and $V\left(T^2 M\right) = \ker d\pi^2$, the vertical subbundle of $T\left(T^2 M\right)$. Its fibres generate the *vertical distribution* V on $T^2 M$ of local dimension $2n$, $V \colon p \in T^2 M \mapsto V(p) \subset T_p\left(T^2 M\right)$, locally spanned by $\left\{ \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^{(2)i}} \right\}$.

In the same way, if the projection $\pi^2_1 \colon T^2 M \to TM$ is given by

$$\left(x^i, y^i, y^{(2)i}\right) \mapsto \left(x^i, y^i\right),$$

then $V_2 := \ker d\pi^2_1$ generates a distribution $V_2 \colon p \in T^2 M \mapsto V_2(p) \subset T_p\left(T^2 M\right)$ of local dimension n, locally spanned by $\left\{ \frac{\partial}{\partial y^{(2)i}} \right\}$.

Then, at any $p \in T^2 M$, there exists a chain of vector spaces

$$V_2(p) \subset V(p) \subset T_p\left(T^2 M\right).$$

Let us consider the $\mathcal{F}\left(T^2 M\right)$-linear mapping $J \colon \mathcal{X}\left(T^2 M\right) \to \mathcal{X}\left(T^2 M\right)$,

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial y^{(2)i}}, \quad J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0, \tag{12}$$

called the *2-tangent structure* on T^2M. J is globally defined on T^2M and $\operatorname{Im} J = V$, $Ker J = V_2$, $J(V) = V_2$.

A *nonlinear connection on* T^2M, [16], is a distribution on T^2M, $N\colon p \in T^2M \to N(p) \subset T_p(T^2M)$, such that

$$T_p(T^2M) = N_0(p) \oplus V(p), \quad \forall p \in T^2M. \tag{13}$$

By setting $N_1(p) := J(N_0(p))$, $\forall p \in T^2M$, we get:

- the *horizontal distribution* $N_0\colon p \mapsto N(p)$;
- the v_1*-distribution* $N_1\colon p \mapsto N_1(p)$;
- the v_2*-distribution* $V_2\colon p \mapsto V_2(p)$, and there holds

$$T_p(T^2M) = N_0(p) \oplus N_1(p) \oplus V_2(p), \quad \forall p \in T^2M.$$

We denote by $h = v_0$, v_1 and v_2 the projectors corresponding to the above distributions.

Let $\mathcal{B}$ denote a local adapted basis to the decomposition (13):

$$\mathcal{B} = \left\{ \delta_{(0)i} := \frac{\delta}{\delta x^i}, \quad \delta_{(1)i} := \frac{\delta}{\delta y^i}, \quad \delta_{(2)i} := \frac{\delta}{\delta y^{(2)i}} \right\},$$

this is, $N_0 = \operatorname{Span}(\delta_{(0)i})$, $N_1 = \operatorname{Span}(\delta_{(1)i})$, $V_2 = Span(\delta_{(2)i})$. The elements of the adapted basis are locally expressed as

$$\begin{aligned}
\delta_{(0)i} &= \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^j} - \underset{(2)}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}} \\
\delta_{(1)i} &= \frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}} \\
\delta_{(2)i} &= \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}.
\end{aligned} \tag{14}$$

With respect to changes of local coordinates on T^2M, induced by changes $(x^i) \mapsto (\tilde{x}^i)$ of local coordinates on the base manifold M, for $\delta_{(\alpha)i}$, $\alpha = 0, 1, 2$, there holds: $\delta_{(\alpha)i} = \dfrac{\partial \widetilde{x}^j}{\partial x^i} \widetilde{\delta}_{(\alpha)j}$.

The dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \left\{dx^i, \delta y^i, \delta y^{(2)i}\right\}$, given by

$$\begin{aligned}
\delta y^{(0)i} &= dx^i, \\
\delta y^i &= dy^i + \underset{(1)}{M}{}^i_j dx^j, \\
\delta y^{(2)i} &= dy^{(2)i} + \underset{(1)}{M}{}^i_j dy^j + \underset{(2)}{M}{}^i_j dx^j.
\end{aligned} \tag{15}$$

The above $\delta y^{(\alpha)i}$, $\alpha = 0, 1, 2$, $i = 1, \ldots, n$, are d-1-forms on T^2M.

The quantities $\underset{(1)}{N}{}^j_i$, $\underset{(2)}{N}{}^j_i$ are called the *coefficients* of the nonlinear connection N, while $\underset{(1)}{M}{}^i_j$ and $\underset{(2)}{M}{}^i_j$ are called its *dual* coefficients. The link between the two sets of coefficients is, [16]:

$$M_{(1)}{}^{i}{}_{j} = N_{(1)}{}^{i}{}_{j}, \quad M_{(2)}{}^{i}{}_{j} = N_{(2)}{}^{i}{}_{j} + N_{(1)}{}^{i}{}_{f} N_{(1)}{}^{f}{}_{j}. \tag{16}$$

In the following, the next result will be very useful to us:

Theorem 2 ([16],[19]-[24]). *1. A transformation of coordinates (3) on the differentiable manifold T^2M implies the following transformation of the dual coefficients of a nonlinear connection*

$$\begin{aligned} \frac{\partial \widetilde{x}^i}{\partial x^k} M_{(1)}{}^{k}{}_{j} &= \widetilde{M}_{(1)}{}^{i}{}_{k} \frac{\partial \widetilde{x}^k}{\partial x^j} + \frac{\partial \widetilde{y}^i}{\partial x^j} \\ \frac{\partial \widetilde{x}^i}{\partial x^k} M_{(2)}{}^{k}{}_{j} &= \widetilde{M}_{(2)}{}^{i}{}_{k} \frac{\partial \widetilde{x}^k}{\partial x^j} + \widetilde{M}_{(1)}{}^{i}{}_{k} \frac{\partial \widetilde{y}^k}{\partial x^j} + \frac{\partial \widetilde{y}^{(2)i}}{\partial x^j}. \end{aligned} \tag{17}$$

2. If on each domain of local chart on T^2M it is given a set of functions $\left(M_{(1)}{}^{i}{}_{j}, M_{(2)}{}^{i}{}_{j} \right)$, such that, with respect to (3), there hold the equalities (17), then there exists on T^2M a unique nonlinear connection N which has as dual coefficients the given set of functions.

In presence of a nonlinear connection, a vector field $X \in \mathcal{X}\left(T^2M\right)$ is represented in the local adapted basis as

$$X = X^{(0)i}\delta_{(0)i} + X^{(1)i}\delta_{(1)i} + X^{(2)i}\delta_{(2)i}, \tag{18}$$

with the three right terms (which are d-vector fields) belonging to the distributions N, N_1 and V_2 respectively.

A 1-form $\omega \in \mathcal{X}^*\left(T^2M\right)$ will be decomposed as

$$\omega = \omega_i^{(0)} dx^i + \omega_i^{(1)} \delta y^i + \omega_i^{(2)} \delta y^{(2)i}. \tag{19}$$

Similarly, a tensor field $T \in \mathcal{T}_s^r\left(T^2M\right)$ can be split with respect to (13) into components, which are d-tensor fields.

In particular, if $\widetilde{c}$: $t \to (x^i(t), y^i(t), y^{(2)i}(t))$ is an extension curve, then its tangent vector field is expressed in the adapted basis as

$$\dot{\widetilde{c}} = \frac{dx^i}{dt}\delta_{(0)i} + \frac{\delta y^i}{dt}\delta_{(1)i} + \frac{\delta y^{(2)i}}{dt}\delta_{(2)i}. \tag{20}$$

Our goal is to give a precise meaning to the equality $v_2(\dot{\widetilde{c}}) = 0$.

5. Berwald linear connection on T^2M

Let $G^i = G^i(x, y)$ be the coefficients of a spray on TM, and

$$N^i{}_j(x, y) = \frac{\partial G^i}{\partial y^j},$$

the coefficients of the induced nonlinear connection (on TM).

Let also

$$L^i{}_{jk}(x,y) = \frac{\partial N^i{}_j}{\partial y^k} = \frac{\partial^2 G^i}{\partial y^j \partial y^k},$$

the local coefficients of the induced Berwald linear connection on TM, [16].

Now, let on T^2M, a linear connection defined by $\underset{(1)}{N}{}^i{}_j = N^i{}_j(x, y^{(1)})$ as above, and arbitrary $\underset{(2)}{N}{}^i{}_j = \underset{(2)}{N}{}^i{}_j(x, y, y^{(2)})$. The *Berwald connection* on T^2M , [8], is the linear connection defined by

$$\begin{aligned} &D_{\delta_{(0)k}} \delta_{(\alpha)j} = L^i{}_{jk} \delta_{(\alpha)i}, \\ &D_{\delta_{(\beta)k}} \delta_{(\alpha)j} = 0, \quad \beta = 1, 2, \ \alpha = 0, 1, 2. \end{aligned} \tag{21}$$

This is, with the notations in [16], the coefficients of the Berwald linear connection are $B\Gamma(N) = (L^i{}_{jk}, 0, 0)$.

For extensions $\widetilde{c}$ to T^2M of curves $c\colon [0.1] \to M$, we can express the v_1 component of the tangent vector field $\dot{\widetilde{c}}$, given by $\dfrac{\delta y^i}{dt}$ (the *geometric acceleration,* [13]) by means of the Berwald covariant derivative:

$$\frac{Dy^i}{dt} := D_{\dot{\widetilde{c}}}\, y^i = \frac{\delta y^i}{dt}, \quad i = 1, \ldots, n. \tag{22}$$

Let $\mathbb{T}$ denote its torsion tensor, and:

$$R^i{}_{jk} = v_1 \mathbb{T}(\delta_{(0)k}, \delta_{(0)j}) = \delta_{(0)k} N^i{}_j - \delta_{(0)j} N^i{}_k,$$

its $v_1(h,h)$ components.

Also, let $\mathbb{R}$ be the curvature tensor; then

$$\begin{aligned} R_j{}^i{}_{kl} &= \delta_{(0)l} L^i{}_{jk} - \delta_{(0)k} L^i{}_{jl} + L^m_{jk} L^i{}_{ml} - L^m_{jl} L^i{}_{mk}, \\ P_j{}^i{}_{kl} &= \delta_{(1)l} L^i{}_{jk} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \end{aligned}$$

where $R_j{}^i{}_{kl}\delta_{(0)i} = h\mathbb{R}(\delta_{(0)l}, \delta_{(0)k})$, $P_j{}^i{}_{kl}\delta_{(0)i} = h\mathbb{R}(\delta_{(1)l}, \delta_{(0)k})$, define its only nonvanishing local components, [16].

Taking into account that $L^i{}_{jk}$ do not depend on $y^{(2)}$ and that $G^i = G^i(x,y)$ are 2-homogeneous in y, it follows:

$$y^j R_j{}^i{}_{kl} = R^i{}_{kl}. \tag{23}$$

From the 2-homogeneity of G^i, we also have

$$P_j{}^i{}_{kl} y^l = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} y^l = 0; \quad P_j{}^i{}_{kl} y^j = P_j{}^i{}_{kl} y^k = 0. \tag{24}$$

6. Jacobi equations for systems with external forces

Let us suppose that we know *a priori* a nonlinear connection on the first order tangent bundle TM, with (1-homogeneous) coefficients $N^i{}_j(x,y) = \dfrac{\partial G^i}{\partial y^j}$, coming from a spray on TM.

Let $c\colon [0,1] \to M$, $t \mapsto x^i(t)$ be a curve on M, such that x^i are solutions for the system of ODE's (10):

$$\frac{\delta \dot{x}^i}{dt} = \frac{1}{2} F^i(x, \dot{x}),$$

where F^i are the components of a d-vector field on M.

Let $\alpha\colon [0,1] \times (-\varepsilon, \varepsilon) \to M$, $(t,u) \mapsto (\alpha^i(t,u))$ denote a variation of c (not necessarily with fixed endpoints): $\alpha^i(t,0) = x^i(t)$, $\forall t \in [0,1]$,

$$y^i = \frac{\partial \alpha^i}{\partial t}|_{u=0} = \frac{dx^i}{dt}$$

the components of the tangent vector field of c and

$$w^i(t) = \frac{\partial \alpha^i}{\partial u}|_{u=0}$$

the components of the deviation vector field attached to the variation α. Let $\widetilde{\alpha}$ denote the following extension of α to the second order tangent bundle T^2M:

$$\widetilde{\alpha}\colon [0,1] \times (-\varepsilon, \varepsilon) \to T^2M, (t,u) \mapsto (\alpha^i(t,u), \ \frac{\partial \alpha^i}{\partial t}(t,u), \frac{1}{2}\frac{\partial^2 \alpha^i}{\partial t^2}(t,u)) \tag{25}$$

and

$$\alpha^i_t = \frac{\partial \alpha^i}{\partial t}, \quad \alpha^i_u = \frac{\partial \alpha^i}{\partial u}.$$

We have:

- $h\left(\dfrac{\partial \widetilde{\alpha}}{\partial t}\right) = \alpha^i_t \delta_{(0)i}$, $h\left(\dfrac{\partial \widetilde{\alpha}}{\partial u}\right) = \alpha^i_u \delta_{(0)i}$;
- $\alpha^i_t(t,0) = y^i(t)$, $\alpha^i_u(t,0) = w^i$, $\forall t \in [0,1]$.

Let us denote $\dfrac{D}{\partial t} = D_{\frac{\partial \widetilde{\alpha}}{\partial t}}$ and $\dfrac{D}{\partial u} = D_{\frac{\partial \widetilde{\alpha}}{\partial u}}$ the covariant derivations with respect to the Berwald connection on T^2M. Then:

$$\begin{aligned}
\frac{D\alpha^i_t}{\partial t} &= \frac{\partial \alpha^i_t}{\partial t} + \ N^i{}_j(\alpha, \alpha_t)\alpha^j_t, \\
\frac{D\alpha^i_t}{\partial u} &= \frac{\partial \alpha^i_t}{\partial u} + \ N^i{}_j(\alpha, \alpha_t)\alpha^j_u, \\
\frac{D\alpha^i_u}{\partial t} &= \frac{\partial \alpha^i_u}{\partial t} + \ N^i{}_j(\alpha, \alpha_t)\alpha^j_u;
\end{aligned} \tag{26}$$

(the covariant derivatives are taken 'with reference vector $\dfrac{\partial \widetilde{\alpha}}{\partial t}$', [5]).

By commuting partial derivatives of α^i, we have $\dfrac{\partial \alpha_t^i}{\partial u} = \dfrac{\partial \alpha_u^i}{\partial t}$, hence that the last two covariant derivatives (26) coincide:

$$\frac{D\alpha_t^i}{\partial u} = \frac{D\alpha_u^i}{\partial t},$$

which is,

$$\frac{D}{\partial u}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) = \frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial u}\right).$$

By applying $D_{\frac{\partial \widetilde{\alpha}}{\partial t}}$ again to the above equality, we get:

$$\frac{D}{\partial t}\frac{D}{\partial u}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) = \frac{D}{\partial t}\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial u}\right). \tag{27}$$

In the left hand side, we can commute covariant derivatives by means of the curvature tensor of D :

$$\frac{D}{\partial t}\frac{D}{\partial u}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) = R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) + \frac{D}{\partial u}\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) + D_{\left[\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right]}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right).$$

But, $\left[\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right]$ is 0, hence the last term in the above relation vanishes and (27) becomes

$$\frac{D}{\partial t}\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial u}\right) = R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right) + \frac{D}{\partial u}\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right). \tag{28}$$

Moreover, at $u = 0$, we have $h\dfrac{\partial \widetilde{\alpha}}{\partial t}|_{u=0} = \alpha_t^i(t,0)\delta_{(0)i} = y^i\delta_{(0)i}$, and by means of (11), we get

$$\frac{D}{\partial t}\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)|_{u=0} = \frac{Dy^i}{\partial t}\delta_{(0)i} = \frac{1}{2}F^i\delta_{(0)i} =: \frac{1}{2}F$$

(where F is a d-vector field on T^2M). Then, (28) becomes

$$\frac{D^2}{\partial t^2}\left(h\frac{\partial \widetilde{\alpha}}{\partial u}|_{u=0}\right) = R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)|_{u=0} + \frac{1}{2}D_uF. \tag{29}$$

At $u = 0$, we also have $h\dfrac{\partial \widetilde{\alpha}}{\partial u} = w^i\delta_{(0)i}$. In local writing, by evaluating

$$R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)$$

and taking into account (24), we obtain

$$R\left(\frac{\partial \widetilde{\alpha}}{\partial t}, \frac{\partial \widetilde{\alpha}}{\partial u}\right)\left(h\frac{\partial \widetilde{\alpha}}{\partial t}\right)|_{u=0} = y^h y^k R_{h\ jk}^{\ i} w^j \delta_{(0)i}.$$

We have thus proved

Proposition 3. *The components of the deviation vector field* $w^i = \frac{\partial \alpha^i}{\partial u}|_{u=0}$ *of the trajectories*

$$\frac{\delta y^i}{dt} = \frac{1}{2} F^i(x,y), \tag{30}$$

satisfy, with respect to the Berwald linear connection on T^2M, *the Jacobi-type equation*

$$\frac{D^2 w^i}{dt^2} = \frac{1}{2} \frac{DF^i}{\partial u}|_{u=0} + y^h y^k R_h{}^i{}_{jk} w^j. \tag{31}$$

The above generalizes the usual Jacobi equation, in the case of mechanical systems with external forces.

7. Nonlinear connection

In natural coordinates, (31) becomes:

$$\begin{aligned} &\frac{d^2 w^i}{dt^2} + \left(2N^i{}_j - \frac{1}{2}\frac{\partial F^i}{\partial y^j} \right) \frac{dw^j}{dt} \\ &\quad + \left(\frac{d}{dt}(N^i{}_j) + N^i{}_k N^k{}_j - y^h y^k R_h{}^i{}_{jk} + L^i{}_{kj} \frac{1}{2} F^k - \frac{1}{2}\frac{\partial F^i}{\partial x^j} \right) w^j = 0. \end{aligned} \tag{32}$$

Taking into account (23), we have $R^i{}_{hjk} y^h = R^i{}_{jk}$. Also, $L^i{}_{kj} = \frac{\partial N^i{}_k}{\partial y^j}$, hence the above equality can be seen as:

$$\begin{aligned} &\frac{d^2 w^i}{dt^2} + \left(2N^i{}_j - \frac{1}{2}\frac{\partial F^i}{\partial y^j} \right) \frac{dw^j}{dt} \\ &\quad + \left(\mathbb{C}(N^i{}_j) + N^i{}_k N^k{}_j - y^k R^{\ i}_{\ jk} + \frac{1}{2}\frac{\partial N^i{}_k}{\partial y^j} F^k - \frac{1}{2}\frac{\partial F^i}{\partial x^j} \right) w^j = 0, \end{aligned}$$

where

$$\mathbb{C} = y^k \frac{\partial}{\partial x^k} + 2y^{(2)k} \frac{\partial}{\partial y^k}.$$

There holds:

Theorem 4. (1) *The quantities*

$$\begin{aligned} \underset{(1)}{M}{}^i{}_j(x,y) &= \frac{1}{2}\left(2N^i{}_j - \frac{1}{2}\frac{\partial F^i}{\partial y^j} \right), \\ \underset{(2)}{M}{}^i{}_j(x,y,y^{(2)}) &= \frac{1}{2}\left(\mathbb{C}(N^i{}_j) + N^i{}_k N^k{}_j - y^k R^{\ i}_{\ jk} \right. \\ &\qquad \left. + \frac{1}{2}\frac{\partial N^i{}_k}{\partial y^j} F^k - \frac{1}{2}\frac{\partial F^i}{\partial x^j} \right) \end{aligned} \tag{33}$$

are the dual coefficients of a nonlinear connection on T^2M.

(2) *With respect to this nonlinear connection, the extensions of deviation vector fields attached to (10) have vanishing* v_2*-components:*

$$\frac{1}{2}\frac{d^2w^i}{dt^2} + \underset{(1)}{M}{}^i{}_j\frac{dw^j}{dt} + \underset{(2)}{M}{}^i{}_j w^j = 0.$$

Proof. 1): In the equation (31), both the left hand side and the right hand side are components of d-vector fields; by a direct computation, it follows that, with respect to local coordinate changes (3) on T^2M, the quantities $\underset{(1)}{M}{}^i{}_j$ and $\underset{(2)}{M}{}^i{}_j$ obey the rules of transformation (17) of the dual coefficients of a nonlinear connection on T^2M.

2): The deviation vector field attached to the variation $\tilde{\alpha}$ in (25) is

$$W = \frac{\partial\tilde{\alpha}}{\partial u}|_{u=0} \equiv \left\{\frac{\partial\alpha^i}{\partial u}\frac{\partial}{\partial x^i} + \frac{\partial}{\partial u}\left(\frac{\partial\alpha^i}{\partial t}\right)\frac{\partial}{\partial y^i} + \frac{1}{2}\frac{\partial}{\partial u}\left(\frac{\partial^2\alpha^i}{\partial t^2}\right)\frac{\partial}{\partial y^{(2)i}}\right\}|_{u=0}$$

$$= w^i\frac{\partial}{\partial x^i} + \frac{dw^i}{dt}\frac{\partial}{\partial y^i} + \frac{1}{2}\frac{d^2w^i}{dt^2}\frac{\partial}{\partial y^{(2)i}}.$$

In the adapted basis $(\delta_{(0)i}, \delta_{(1)i}, \delta_{(2)i})$, this yields:

$$W = w^i\delta_{(0)i} + \frac{\delta w^i}{dt}\delta_{(1)i} + \frac{\delta w^{(2)i}}{dt}\delta_{(2)i},$$

where $\frac{\delta w^i}{dt} = \frac{dw^i}{dt} + \underset{(1)}{M}{}^i{}_j(x,y)w^j$ and

$$\frac{\delta w^{(2)i}}{dt} = \frac{1}{2}\frac{d^2w^i}{dt^2} + \underset{(1)}{M}{}^i{}_j(x,y)\frac{dw^j}{dt} + \underset{(2)}{M}{}^i{}_j(x,y,y^{(2)})w^j.$$

Taking into account (33), the Jacobi equation (32) is re-expressed as: □

$$\frac{\delta w^{(2)i}}{dt} = 0.$$

In presence of the above nonlinear connection, the extension W to T^2M of any Jacobi field on M, corresponding to trajectories (10) in presence of external forces, belongs to the $N_0 \oplus N_1$ distribution.

8. Deviations of geodesics

Let us examine the particular case when $F = 0$. Let TM be endowed with a spray with coefficients $G^i = G^i(x,y)$ and $N^i{}_j = \frac{\partial G^i}{\partial y^j}$, the coefficients of the associated nonlinear connection on TM.

If $F = 0$, then we deal with deviations of autoparallel curves (called *geodesics*)

$$\frac{\delta y^i}{dt} = 0.$$

We get

$$\underset{(1)}{M}{}^i{}_j = N^i{}_j,$$

$$\underset{(2)}{M}{}^i{}_j = \frac{1}{2}(\mathbb{C}(N^i{}_j) + N^i{}_k N^k_j - y^j R^i{}_{jk});$$

taking into account that, in our approach, $\underset{(1)}{M}{}^i{}_j$ do not depend on $y^{(2)}$, we notice that, in the case $F = 0$, our nonlinear connection only differs by the term $-y^j R^i{}_{jk}$ from Miron's one (1), [16].

Remark 5. Along an extension curve $\widetilde{c}\colon [0,1] \to T^2M$, $t \mapsto (x^i(t), y^i(t) = \dot{x}^i(t), y^{(2)i}(t) = \frac{1}{2}\ddot{x}^i(t))$ there hold the equalities

$$\frac{\delta y^i}{dt} = \frac{Dy^i}{dt}, \quad \frac{\delta y^{(2)i}}{dt} = \frac{D^2 y^i}{dt^2},$$

where $\frac{D}{dt}$ denotes the covariant derivative associated to the Berwald connection on T^2M. For these curves, taking into account the equalities $y^j y^k R^i{}_{jk} = 0$ (which can be obtained by direct calculation), it follows that, with the assumptions made at the beginning of this section, $\frac{\delta y^i}{dt}$ and $\frac{\delta y^{(2)i}}{dt}$ have the same values as those obtained for the connection (1). Still, along general curves γ on T^2M, the value of $v_2(\dot{\gamma})$ does no longer coincide with that one obtained with respect to (1).

Remark 6. Also, for a vector field w along the projection c of $\widetilde{c}$ onto M, we have

$$\frac{\delta w^i}{dt} = \frac{Dw^i}{dt}.$$

Conclusions:

(1) c is a geodesic if and only if its extension to T^2M is horizontal.
(2) For a vector field w along a geodesic c on M, we have:
  (a) $\frac{\delta w^i}{dt} = 0$, if and only if w is parallel along $\dot{c} = y$.
  (b) $\frac{\delta w^{(2)i}}{dt} = 0$ if and only if w is a Jacobi field along c.

In the case $F = 0$, we should mention some related results and approaches:

In the geometry of TM: In the case when the base manifold M is endowed with a linear connection ∇, a linear connection on the tangent bundle TM, with similar properties to those of (33) is given by the *complete lift* ∇^C *of* ∇ (cf. [28] and [10]). Namely, in the two cited monographs, it is shown that, if a curve $\bar{\sigma}\colon [0,1] \to TM$, $t \mapsto (x^i(t), w^i(t))$ is a geodesic with respect to ∇^C, then its projection $\sigma\colon t \mapsto (x^i(t))$ onto M is a geodesic with respect to ∇ and $X(t) = w^i(t)\dfrac{\partial}{\partial x^i}$ is a Jacobi field along σ.

In the geometry of T^2M: In presence of a linear connection ∇ on M, C. Dodson and M. Radivoiovici, [11] built a covariant derivation law $\bar{\nabla} : \mathcal{X}(M) \times \Gamma(T^2M) \to \Gamma(T^2M)$ for sections of the second order tangent bundle (regarded as a vector bundle over M) and used it in order to define a nonlinear connection in the frame bundle of order 2 $L^{(2)}M$. In the case when ∇ is torsion-free, the covariant derivative $\bar{\nabla}_v X$, where $v = \dfrac{\partial \alpha}{\partial u}|_{u=0}$, and $X \equiv \left(\dfrac{\partial \alpha}{\partial t}, \dfrac{D}{dt}\dfrac{\partial \alpha}{\partial t}\right)$ (with our notations in Section 6) would yield our $\left(\dfrac{\delta w^i}{dt}, \dfrac{\delta w^{(2)i}}{dt}\right)$. Still, in the cited paper, it is not established any link between the defined connection and the Jacobi equation on M.

The novelty of our approach consists in relating the v_2-distribution on T^2M to deviations of geodesics of the base manifold.

9. External forces in Finsler-locally Minkowskian spaces

Another interesting particular case is that of Finsler-locally Minkowskian spaces (whose geodesics are straight lines). Let $(M, L(y))$ be a Finsler-locally Minkowskian space, [2], [5].

Then, ${N^i}_j = 0$, ${L^i}_{jk} = 0$ (for the Berwald connection), [2], [5]. In presence of an external force field, the evolution equations of a mechanical system will take the form

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}F^i(x, \dot{x}). \tag{34}$$

In this case, with the above notations, our nonlinear connection is given by

$$\underset{(1)}{M^i}{}_j = -\frac{1}{4}\frac{\partial F^i}{\partial y^j},$$

$$\underset{(2)}{M^i}{}_j = -\frac{1}{4}\frac{\partial F^i}{\partial x^j}.$$

This is, deviations of the evolution curves (34) can be written simply:

$$2\frac{\delta w^{(2)i}}{dt} \equiv \frac{d^2w^i}{dt^2} - \frac{1}{2}\frac{\partial F^i}{\partial y^j}\frac{dw^j}{dt} - \frac{1}{2}\frac{\partial F^i}{\partial x^j}w^j = 0.$$

The result holds valid for any globally defined system of ordinary differential equations of order 2 on M, of the form (34).

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