

MAXIMAL (C, α, β) OPERATORS OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to prove that for the boundedness of the maximal operator $\sigma_*^{\alpha, \beta}$ from the Hardy space $H_p(I^2)$ to the space $L_p(I^2)$ the assumption $p > \max\{1/(\alpha + 1), 1/(\beta + 1)\}$ is essential.

We denote the set of non-negative integers by $\mathbf{N}$. For a set $X \neq \emptyset$ let X^2 be its Cartesian product $X \times X$ taken with itself. By a dyadic interval in $I := [0, 1)$ we mean one of the form $[l2^{-k}, (l+1)2^{-k})$ for some $k \in \mathbf{N}$, $0 \leq l < 2^k$. Given $k \in \mathbf{N}$ and $x \in [0, 1)$, let $I_k(x)$ denote the dyadic interval of length 2^{-k} which contains the point x . The Cartesian product of two dyadic intervals is said to be a rectangle. Clearly, the dyadic rectangle of area $2^{-n} \times 2^{-m}$ containing $(x^1, x^2) \in I^2$ is given by $I_{n,m}(x^1, x^2) := I_n(x^1) \times I_m(x^2)$. We also use the notation $\text{mes}(A)$ for the Lebesgue measure of any measurable set A .

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2), \\ -1, & \text{if } x \in [1/2, 1), \end{cases}$$
$$r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1 \text{ and } x \in [0, 1).$$

Let $w_0, w_1, \dots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if

$$n = 2^{n_1} + \dots + 2^{n_r}$$

is a positive integer with $n_1 > n_2 > \dots > n_r$ then

$$w_n(x) = r_{n_1}(x) \cdots r_{n_r}(x).$$

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The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}), \\ 0, & \text{if } x \in [2^{-n}, 1). \end{cases}$$

The Kronecker product $(w_{n,m} : n, m \in \mathbf{N})$ of two Walsh systems is said to be the two-dimensional Walsh system. Thus

$$w_{n,m}(x^1, x^2) := w_n(x^1) w_m(x^2).$$

The partial sums of the two-dimensional Walsh-Fourier series are defined as follows:

$$S_{n,m}f(x^1, x^2) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \hat{f}(i, j) w_{i,j}(x^1, x^2),$$

where the number

$$\hat{f}(i, j) = \int_I f(u^1, u^2) w_{i,j}(u^1, u^2) du^1 du^2$$

is said to be the (i, j) th Walsh-Fourier coefficient of the function f .

The norm (or quasinorm) of the space $L_p(I^2)$ is defined by

$$\|f\|_p := \left(\int_{I^2} |f(x^1, x^2)|^p dx^1 dx^2 \right)^{1/p} \quad (0 < p < +\infty).$$

The σ -algebra generated by the dyadic rectangles $\{I_{n,m}(x^1, x^2) : x, y \in I\}$ will be denoted by $F_{n,m}$ ($n, m \in \mathbf{N}$), more precisely,

$$F_{n,m} = \sigma \{ [k2^{-n}, (k+1)2^{-n}) \times [l2^{-m}, (l+1)2^{-m}) : 0 \leq k < 2^n, 0 \leq l < 2^m \},$$

where $\sigma(A)$ denotes the σ -algebra generated by an arbitrary set system A .

Denote by $f = (f^{(n,m)}, n \in \mathbf{N})$ two-parameter martingale with respect to $(F_{n,m}, n, m \in \mathbf{N})$ (for details see, e.g. [6, 9]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n,m \in \mathbf{N}} |f^{(n,m)}|.$$

In case $f \in L_1(I^2)$, the maximal function can also be given by

$$f^*(x^1, x^2) = \sup_{n,m \in \mathbf{N}} \frac{1}{\text{mes}(I_n(x^1) \times I_m(x^2))} \left| \int_{I_n(x^1) \times I_m(x^2)} f(u, v) du dv \right|,$$

$(x^1, x^2) \in I^2$.

For $0 < p < \infty$ the Hardy martingale space $H_p(I^2)$ consists all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(I^2)$ then it is easy to show that the sequence $(S_{2^n, 2^m}(f) : n, m \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n,m)} : n, m \in \mathbf{N})$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i, j) = \lim_{k, l \rightarrow \infty} \int_{I^2} f^{(k, l)}(x^1, x^2) w_i(x^1) w_j(x^2) dx^1 dx^2.$$

The Walsh-Fourier coefficients of the function $f \in L_1(I^2)$ are the same as the ones of the martingale $(S_{2^n, 2^m}(f) : n, m \in \mathbf{N})$ obtained from the function f .

The (C, α, β) means of the two-dimensional Walsh-Fourier series of the martingale f is given by

$$\sigma_{n,m}^{\alpha,\beta}(f, x^1, x^2) = \frac{1}{A_{n-1}^\alpha} \frac{1}{A_{m-1}^\beta} \sum_{i=1}^n \sum_{j=1}^m A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j} f(x^1, x^2),$$

where

$$A_n^\alpha := \frac{(1 + \alpha) \dots (n + \alpha)}{n!}$$

for any $n \in \mathbf{N}, \alpha \neq -1, -2, \dots$. It is known ([10]) that $A_n^\alpha \sim n^\alpha$.

For the martingale f we consider the maximal operator

$$\sigma_*^{\alpha,\beta} f = \sup_{n,m} |\sigma_{n,m}^{\alpha,\beta}(f, x^1, x^2)|.$$

The (C, α) kernel defined by

$$K_n^\alpha(x) := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^n A_{n-k}^{\alpha-1} D_k(x).$$

In the one-dimensional case, Fine [1] proved that the (C, α) means $\sigma_n^\alpha f$ of a function $f \in L(I)$ converge a.e. to f as $n \rightarrow \infty$. The maximal operator $\sigma_*^\alpha f := \sup_n |\sigma_n^\alpha f|$ ($0 < \alpha < 1$) of the (C, α) means of the Walsh-Paley Fourier series was investigated by Weisz [8]. In his paper Weisz proved the boundedness of $\sigma_*^\alpha : H_p \rightarrow L_p$ when $p > 1/(1 + \alpha)$. The author [3] showed that in Theorem of Weisz the assumption $p > 1/(\alpha + 1)$ is essential. In particular, we proved that the maximal operator σ_*^α of the (C, α) means of the Walsh-Paley Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I)$ to the space $L_{1/(\alpha+1)}(I)$.

For double Walsh-Fourier series it is known [5] that the (C, α, β) means $\sigma_{n,m}^{\alpha,\beta} f \rightarrow f$ in L_p norm as $n, m \rightarrow \infty$ whenever $f \in L_p(I^2)$ for some $1 \leq p < \infty$.

On the other hand, in 1992 Móricz, Schipp and Wade [4] proved with respect to the Walsh-Paley system that

$$\sigma_{n,m}f = \frac{1}{nm} \sum_{i=1}^n \sum_{k=1}^m S_{i,k}(f) \rightarrow f$$

a.e. for each $f \in L \log^+ L([0, 1]^2)$, when $\min\{n, m\} \rightarrow \infty$. In 2000 Gát proved [2] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0, +\infty) \rightarrow [0, +\infty)$ be a measurable function with property $\lim_{t \rightarrow \infty} \delta(t) = 0$. Gát proved [2] the existence of a function $f \in L^1(I^2)$ such that $f \in L \log^+ L\delta(L)$, and $\sigma_{n,m}f$ does not converge to f a.e. as $\min\{n, m\} \rightarrow \infty$. That is, the maximal convergence space for the $(C, 1)$ means of two-dimensional partial sums is $L \log^+ L(I^2)$. Weisz [7] investigated the maximal operator of (C, α, β) means of two-dimensional Walsh-Fourier series and proved that the maximal operator $\sigma_*^{\alpha, \beta} f$ is bounded from $H_p(I^2)$ to $L_p(I^2)$ if $1/(1+\alpha), 1/(1+\beta) < p < \infty$. In [7] Weisz conjectured that for the boundedness of the maximal operator $\sigma_*^{\alpha, \beta}$ from the Hardy space $H_p(I)$ to the space $L_p(I)$ the assumption $p > 1/(\alpha+1), 1/(1+\beta)$ is essential. We give answer to the question and prove that the maximal operator $\sigma_*^{\alpha, \beta}$ of the (C, α, β) ($0 < \alpha \leq \beta \leq 1$) means of the two-dimensional Walsh-Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I^2)$ to the space $L_{1/(\alpha+1)}(I^2)$. The following is true.

Theorem 1. *Let $0 < \alpha \leq \beta \leq 1$. Then the maximal operator $\sigma_*^{\alpha, \beta}$ of the (C, α, β) means of the two-dimensional Walsh-Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I^2)$ to the space $L_{1/(\alpha+1)}(I^2)$.*

In order to prove Theorem 1 we need the following lemma.

Lemma 1. ([3]) *Let $n \in \mathbf{N}$ and $0 < \alpha \leq 1$. Then*

$$\int_I \max_{1 \leq N \leq 2^n} (A_{N-1}^\alpha |K_N^\alpha(x)|)^{1/(\alpha+1)} dx \geq c(\alpha) \frac{n}{\log(n+2)}.$$

Proof of Theorem 1. Let

$$f_n(x^1, x^2) := [D_{2^{n+1}}(x^1) - D_{2^n}(x^1)] w_{2^n-1}(x^2).$$

Since

$$\begin{aligned} \widehat{f}_n(\nu, \mu) &= \int_I [D_{2^{n+1}}(u^1) - D_{2^n}(u^1)] w_\nu(u^1) du^1 \int_I w_{2^n-1}(u^2) w_\mu(u^2) du^2 \\ &= \begin{cases} 1, & \text{if } \nu = 2^n, \dots, 2^{n+1} - 1, \mu = 2^n - 1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

we can write

$$(1) \quad S_{i,j} f_n(x^1, x^2) = \sum_{\nu=0}^{i-1} \widehat{f}_n(\nu, 2^n - 1) w_\nu(x^1) w_{2^n-1}(x^2)$$

$$= \begin{cases} [D_i(x^1) - D_{2^n}(x^1)] w_{2^{n-1}}(x^2), & \text{if } i = 2^n + 1, \dots, 2^{n+1} - 1, j \geq 2^n, \\ f_n(x^1, x^2), & \text{if } i \geq 2^{n+1}, j \geq 2^n, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$f_n^*(x^1, x^2) = \sup_{i,j} |S_{2^i, 2^j} f_n(x^1, x^2)| = |f_n(x^1, x^2)|,$$

$$(2) \quad \|f_n\|_{H_p} = \|f_n^*\|_p = \|D_{2^n}\|_p = 2^{n(1-1/p)}.$$

Let $1 \leq N < 2^n$. Then from (1) we obtain

$$\begin{aligned} \sigma_{2^n+N, 2^{n+1}}^\alpha f_n(x^1, x^2) &= \\ &= \frac{1}{A_{2^n+N-1}^\alpha} \frac{1}{A_{2^{n+1}-1}^\beta} \left| \sum_{i=1}^{2^n+N} \sum_{j=1}^{2^{n+1}} A_{2^n+N-i}^{\alpha-1} A_{2^{n+1}-j}^{\beta-1} S_{i,j} f_n(x^1, x^2) \right| \\ &= \frac{1}{A_{2^n+N-1}^\alpha} \frac{1}{A_{2^{n+1}-1}^\beta} \left| \sum_{i=2^n+1}^{2^n+N} \sum_{j=2^n}^{2^{n+1}} A_{2^n+N-i}^{\alpha-1} A_{2^{n+1}-j}^{\beta-1} S_{i,j} f_n(x^1, x^2) \right| \\ &= \frac{1}{A_{2^n+N-1}^\alpha} \frac{1}{A_{2^{n+1}-1}^\beta} \times \\ &\quad \times \left| \sum_{i=2^n+1}^{2^n+N} \sum_{j=2^n}^{2^{n+1}} A_{2^n+N-i}^{\alpha-1} A_{2^{n+1}-j}^{\beta-1} [D_i(x^1) - D_{2^n}(x^1)] w_{2^{n-1}}(x^2) \right| \\ &\geq \frac{c(\alpha, \beta)}{2^{n\alpha} 2^{n\beta}} \left| \sum_{i=1}^N A_{N-i}^{\alpha-1} [D_{i+2^n}(x^1) - D_{2^n}(x^1)] \right| \left| \sum_{j=0}^{2^n} A_{2^n-j}^{\beta-1} \right| \\ &\geq \frac{c(\alpha, \beta)}{2^{n\alpha}} \left| \sum_{i=1}^N A_{N-i}^{\alpha-1} [D_{i+2^n}(x^1) - D_{2^n}(x^1)] \right| \\ &= \frac{c(\alpha, \beta)}{2^{n\alpha}} \left| \sum_{i=1}^N A_{N-i}^{\alpha-1} D_i(x^1) \right|. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma_*^{\alpha, \beta} f_n(x^1, x^2) &\geq \sup_{1 \leq N \leq 2^n} |\sigma_{2^n+N, 2^{n+1}}^\alpha f_n(x^1, x^2)| \\ &\geq \frac{c(\alpha, \beta)}{2^{n\alpha}} \sup_{1 \leq N \leq 2^n} \left| \sum_{i=1}^N A_{N-i}^{\alpha-1} D_i(x^1) \right|. \end{aligned}$$

Then from Lemma 1 and (2) we get

$$\frac{\|\sigma_*^{\alpha, \beta} f_n\|_{1/(\alpha+1)}}{\|f_n\|_{H_{1/(\alpha+1)}}} \geq \frac{c(\alpha, \beta)}{2^{n\alpha} 2^{-n\alpha}} \left(\int_I \sup_{1 \leq N \leq 2^n} (A_{N-1}^\alpha |K_N^\alpha(x)|)^{1/(\alpha+1)} dx \right)^{\alpha+1}$$

$$\geq c(\alpha, \beta) \left(\frac{n}{\log(n+2)} \right)^{\alpha+1} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Theorem 1 is proved. □

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