

SOME GENERAL OSTROWSKI-GRÜSS TYPE INEQUALITIES

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ABSTRACT. Some new generalizations of Ostrowski-Grüss type inequalities for functions of Lipschitzian type and probability distribution inequalities are established.

1. INTRODUCTION

Definition 1.1. The function $f: [a, b] \rightarrow \mathbb{R}$ is said to be (l, L) -Lipschitzian on $[a, b]$ if

$$l(x_2 - x_1) \leq f(x_2) - f(x_1) \leq L(x_2 - x_1), \quad \text{for } a \leq x_1 \leq x_2 \leq b,$$

where $l, L \in \mathbb{R}$ with $l < L$.

Liu [4] generalizes the results in [2] and [5] to functions which are L -Lipschitzian and (l, L) -Lipschitzian respectively as follows.

Theorem L. Let $f: [a, b] \rightarrow \mathbb{R}$ be (l, L) -Lipschitzian on $[a, b]$. Then we have

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) + \left(x - \frac{a+b}{2} \right) S \right| \leq \frac{b-a}{2} (S-l)$$

and

$$(1.2) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) + \left(x - \frac{a+b}{2} \right) S \right| \leq \frac{b-a}{2} (L-S)$$

for all $x \in [a, b]$, where $S = (f(b) - f(a))/(b-a)$.

Recently there are many versions of the Ostrowski-Grüss type inequalities (see [1, 2, 3, 4, 5]). In the present paper, we will give some general Ostrowski-Grüss type inequalities in view of the probability theory.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Ostrowski-Grüss type inequalities; probability distribution.

2. MAIN RESULTS

The following lemma is easy from the elementary knowledge of probability theory.

Lemma 2.1. *Let X be a random variable with uniform distribution on the support interval $[a, b]$, i.e., the probability density function of X is equal to $(b - a)^{-1}$, $x \in [a, b]$ and zero elsewhere. Accordingly, let us denote $\mathbb{E}X$ the mathematical expectation of random variable X . Then for any $x \in [a, b]$, we have*

$$\mathbb{E}(X - x)1_{X \geq x} = \frac{(b - x)^2}{2(b - a)}$$

and

$$\mathbb{E}(x - X)1_{X < x} = \frac{(x - a)^2}{2(b - a)},$$

where 1_A denotes the indicator function of the set $A \subset [a, b]$.

From the above key lemma, we will give a more precise version of Theorem L.

Theorem 2.2. *Under the assumptions of Theorem L, we have*

$$\begin{aligned} (2.1) \quad & (l - S) \frac{(b - x)^2}{2(b - a)} + (S - L) \frac{(x - a)^2}{2(b - a)} \\ & \leq \frac{1}{b - a} \int_a^b f(t) dt - f(x) + \left(x - \frac{a + b}{2} \right) S \\ & \leq (L - S) \frac{(b - x)^2}{2(b - a)} + (S - l) \frac{(x - a)^2}{2(b - a)}, \end{aligned}$$

for all $x \in [a, b]$.

Proof. Let X be a random variable with uniform distribution on the support interval $[a, b]$. Then it is easy to see that

$$\frac{1}{b - a} \int_a^b f(t) dt - f(x) = \mathbb{E}(f(X) - f(x))$$

and

$$x - \frac{a + b}{2} = \mathbb{E}(x - X).$$

Thus, we have

$$\begin{aligned} & \frac{1}{b - a} \int_a^b f(t) dt - f(x) + \left(x - \frac{a + b}{2} \right) S \\ & = \mathbb{E}(f(X) - f(x)) + S\mathbb{E}(x - X) = \mathbb{E}[(f(X) - f(x)) - S(X - x)]1_{X \geq x} \\ & \quad + \mathbb{E}[(f(X) - f(x)) - S(X - x)]1_{X < x}. \end{aligned}$$

Furthermore, by Definition 1.1 and Lemma 2.1, it follows that

$$\begin{aligned}
 (l-S)\frac{(b-x)^2}{2(b-a)} + (S-L)\frac{(x-a)^2}{2(b-a)} &= \\
 &= (l-S)\mathbb{E}(X-x)1_{X \geq x} + (S-L)\mathbb{E}(x-X)1_{X < x} \\
 &\leq \mathbb{E}[(f(X) - f(x)) - S(X-x)]1_{X \geq x} + \\
 &\quad + \mathbb{E}[(f(X) - f(x)) - S(X-x)]1_{X < x} \\
 &\leq (L-S)\mathbb{E}(X-x)1_{X \geq x} + (S-l)\mathbb{E}(x-X)1_{X < x} \\
 &= (L-S)\frac{(b-x)^2}{2(b-a)} + (S-l)\frac{(x-a)^2}{2(b-a)}
 \end{aligned}$$

which implies the inequality (2.1). $\square$

Corollary 2.3. *Under the assumptions of Theorem L, we have*

$$(2.2) \quad \left| \frac{1}{b-a} \int_a^b f(t)dt - f(x) + \left(x - \frac{a+b}{2}\right) S \right| \leq M \frac{(b-a)}{2},$$

for all $x \in [a, b]$, where $M = \max\{(L-S), (S-l)\}$.

Proof. Let $g(x) = A(b-x)^2 + B(x-a)^2$, where A, B are two positive constants. Then it is easy to check that the maximum point of $g(x)$ is at $x = a$ or $x = b$. From this fact and Theorem 2.2, the desired inequality (2.2) is obtained. $\square$

Corollary 2.4. *Under the assumptions of Theorem L, we have*

$$(2.3) \quad \left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)}{8} (L-l).$$

Corollary 2.5. *Under the assumptions of Theorem L, we have*

$$(2.4) \quad -(S-l)\frac{(b-a)}{2} \leq \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(a)+f(b)}{2} \leq (L-S)\frac{(b-a)}{2}$$

and

$$(2.5) \quad -(L-S)\frac{(b-a)}{2} \leq \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(a)+f(b)}{2} \leq (S-l)\frac{(b-a)}{2}.$$

3. INEQUALITIES IN PROBABILITY DISTRIBUTION

In this section we will give some results for probability distribution.

Lemma 3.1. *Let X be a random variable on $[a, b]$ and $\mathbb{E}(X)$ is the expectation of X . Then we have the following equality,*

$$\begin{aligned}
 (3.1) \quad & \mathbb{P}(X \leq x) - \frac{1}{b-a} \left(x - \frac{a+b}{2} \right) - \frac{b - \mathbb{E}(X)}{b-a} = \\
 & = \mathbb{P}(X \leq x) + \frac{\mathbb{E}X - x}{b-a} - \frac{1}{2} \\
 & = \frac{\int_a^x \mathbb{P}(t \leq X \leq x) dt - \int_x^b \mathbb{P}(x < X < t) dt}{b-a} + \frac{a-x}{b-a} + \frac{1}{2}.
 \end{aligned}$$

Proof. Noting the following facts,

$$\begin{aligned}
 & \mathbb{P}(X \leq x) + \frac{\mathbb{E}X}{b-a} = \\
 & = \frac{\int_a^b \mathbb{P}(X \leq x) dt + \mathbb{E}(\int_a^X dt) + a}{b-a} \\
 & = \frac{\int_a^b \mathbb{P}(X \leq x) dt + \int_a^b \mathbb{P}(X \geq t) dt + a}{b-a} \\
 & = \frac{\int_a^x [\mathbb{P}(X \leq x) + \mathbb{P}(X \geq t)] dt + \int_x^b [\mathbb{P}(X \leq x) + \mathbb{P}(X \geq t)] dt + a}{b-a} \\
 & = \frac{x-a + \int_a^x \mathbb{P}(t \leq X \leq x) dt + b-x - \int_x^b \mathbb{P}(x < X < t) dt + a}{b-a} \\
 & = \frac{b-a + \int_a^x \mathbb{P}(t \leq X \leq x) dt - \int_x^b \mathbb{P}(x < X < t) dt + a}{b-a}
 \end{aligned}$$

and

$$\mathbb{P}(X \leq x) - \frac{1}{b-a} \left(x - \frac{a+b}{2} \right) - \frac{b - \mathbb{E}X}{b-a} = \mathbb{P}(X \leq x) + \frac{\mathbb{E}X - x}{b-a} - \frac{1}{2},$$

we have

$$\begin{aligned}
 & \mathbb{P}(X \leq x) + \frac{\mathbb{E}X - x}{b-a} - \frac{1}{2} \\
 & = \frac{\int_a^x \mathbb{P}(t \leq X \leq x) dt - \int_x^b \mathbb{P}(x < X < t) dt}{b-a} + \frac{a-x}{b-a} + \frac{1}{2}.
 \end{aligned}$$

□

Corollary 3.2. *Under the assumptions of Lemma 3.1, we have the following inequalities*

$$\begin{aligned}
 & -\frac{(b-x)\mathbb{P}(X > x)}{b-a} + \frac{a-x}{b-a} + \frac{1}{2} \leq \\
 (3.2) \quad & \leq \frac{\int_a^x \mathbb{P}(t \leq X \leq x)dt - \int_x^b \mathbb{P}(x < X < t)dt}{b-a} + \frac{a-x}{b-a} + \frac{1}{2} \\
 & \leq \frac{(x-a)\mathbb{P}(X \leq x)}{b-a} + \frac{a-x}{b-a} + \frac{1}{2}.
 \end{aligned}$$

Furthermore

$$\left| \frac{\int_a^x \mathbb{P}(t \leq X \leq x)dt - \int_x^b \mathbb{P}(x < X < t)dt}{b-a} + \frac{a-x}{b-a} + \frac{1}{2} \right| \leq \frac{1}{2}.$$

Corollary 3.3. *Under the assumptions of Lemma 3.1, we have*

$$(3.3) \quad \left| \mathbb{E}(X) - \frac{a+b}{2} \right| \leq \frac{b-a}{2}.$$

Proof. We set $x = a$ or $x = b$ in (3.1) and by Corollary 3.2 to get the desired result. $\square$

Corollary 3.4. *Under the assumptions of Lemma 3.1, we have*

$$(3.4) \quad -\frac{\mathbb{P}(X > \frac{a+b}{2})}{2} \leq \mathbb{P}\left(X \leq \frac{a+b}{2}\right) - \frac{b - \mathbb{E}(X)}{b-a} \leq \frac{\mathbb{P}(X \leq \frac{a+b}{2})}{2}.$$

In particular, assume that m is the median of the random variable X , i.e., $\mathbb{P}(X \leq m) \geq 1/2$ and $\mathbb{P}(X \geq m) \geq 1/2$, then we have

$$\left| \mathbb{P}(X \leq m) - \frac{1}{b-a} \left(m - \frac{a+b}{2} \right) - \frac{b - \mathbb{E}(X)}{b-a} \right| \leq \frac{1}{4}.$$

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Received March 24, 2008.

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