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ON CLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

K. AL SHAQSI AND M. DARUS

Abstract. Let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

defined on the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. In this paper we define a subclass of α-uniform starlike and convex functions by using the generalized Ruscheweyh derivatives operator introduced by authors in [9]. Several properties to this class are obtained.

1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions of the form $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, defined on the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{U}$. Let $S^*(\beta)$ and $C(\beta)$ be the classes of functions respectively starlike of order β and convex of order β, $(0 \le \beta < 1)$. Finally, let $\mathcal{T}$ be the subclass of $\mathcal{S}$, consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k. \tag{1}$$

A function $f \in \mathcal{T}$ is called a function with negative coefficients. In this present paper, we study the following class of function:

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Definition 1.1. For $0 \leq \beta < 1$, $\alpha \geq 0$, $n \in \mathbb{N}_0$ and $\lambda \geq 0$, we let $M_\lambda^n(\alpha,\beta)$, consist of functions $f \in \mathcal{T}$ satisfying the condition

$$\Re\left\{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}\right\} > \alpha\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1\right| + \beta \tag{2}$$

where D_λ^n denote the operator introduced by authors [9] and given by

$$D_\lambda^n f(z) = \frac{z(z^{n-1}D_\lambda f(z))^{(n)}}{n!}, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Note that if f is given by (1), then we see that

$$D_\lambda^n f(z) = z - \sum_{k=2}^{\infty} \big[1 + \lambda(k-1)\big] C(n,k)|a_k| z^k,$$

where $\lambda \geq 0$, $n \in \mathbb{N}_0$ and $C(n,k) = \binom{k+n-1}{n}$.

The family $M_\lambda^n(\alpha,\beta)$ is of special interest for it contains many well known, as well as new, classes of analytic univalent functions. In particular $M_\lambda^1(\alpha,\beta) \equiv \mathcal{U}(k,\lambda,\beta)$ is the class of α-uniformly convex function introduced and studied by Shanmugam et al. [8]. The classes $M_1^0(\alpha,0) \equiv \alpha\text{-}ST$ $M_1^1(\alpha,0) \equiv \alpha\text{-}UCV$ is respectively, the classes of α-uniformly starlike function and α-uniformly convex function introduced and studied by Kanas and Wisniowska [5, 4]. The classes $M_0^0(0,\beta) \equiv \mathcal{T}^*(\beta)$ and $M_0^1(0,\beta) \equiv \mathcal{T}C(\beta)$ is respectively the classes of starlike functions of order β and classes of convex functions of order β studied by Silverman [10]. Also, we note that the class $M_1^0(1,1) \equiv UCV$ was studied by Rønning [6]. Finally, we remark that Goodman introduced the concept of uniformly starlike function and of uniformly convex function in [3] and proved some properties for such functions in [3] and [2].

In this paper we provide necessary and sufficient conditions, coefficient bounds, extreme points, radius of close-to-convexity, starlikeness and convexity for functions in $M_\lambda^n(\alpha,\beta)$. Inclusion theorem involving Hadamard products, convolution and integral operator are also obtained.

2. Characterization

We employ the technique adopted by Aqlan et al. [1] to find the coefficient estimates for our class.

Theorem 2.1. *let f given by (1) then, $f \in M_\lambda^n(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty} \big[k - \beta + \alpha(k-1)\big]\big[1 + \lambda(k-1)\big] C(n,k)|a_k| \leq (1-\beta), \tag{3}$$

where $\alpha, \lambda \geq 0$, $0 \leq \beta < 1$ and $n \in \mathbb{N}_0$. The result is sharp.

Proof. We have $f \in M_\lambda^n(\alpha,\beta)$ if and only if the condition (2) is satisfied. Upon the fact that

$$\Re(w) > \alpha|w-1| + \beta \Leftrightarrow \Re\Big\{w\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta}\big)\Big\} > \beta, \quad -\pi \le \theta < \pi.$$

Equation (2) may be written as

$$(4) \quad \Re\left\{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta}\right\} = \Re\left\{\frac{z(D_\lambda^n f(z))'\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta} D_\lambda^n f(z)}{D_\lambda^n f(z)}\right\} > \beta.$$

Now, we let

$$A(z) = z(D_\lambda^n f(z))'\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta} D_\lambda^n f(z), \quad B(z) = D_\lambda^n f(z).$$

Then (4) is equivalent to $|A(z)+(1-\beta)B(z)| > |A(z)-(1+\beta)B(z)|$ for $0 \le \beta < 1$. For $A(z)$ and $B(z)$ as above, we have

$$|A(z)+(1-\beta)B(z)| \ge (2-\beta)|z| - \sum_{k=2}^{\infty}\big[k+1-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)|a_k||z|^k,$$

and similarly

$$|A(z)-(1+\beta)B(z)| \le \beta|z| - \sum_{k=2}^{\infty}\big[k-1-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)|a_k||z|^k.$$

Therefore,

$$|A(z)+(1-\beta)B(z)| - |A(z)-(1+\beta)B(z)| \ge 2(1-\beta) - 2\sum_{k=2}^{\infty}\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)|a_k|,$$

or $\sum_{k=2}^{\infty}\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)|a_k| \le (1-\beta)$, which yields (3).

On the other hand, we must have $\Re\left\{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)}\big(1+\alpha e^{i\theta}\big) - \alpha e^{i\theta}\right\} > \beta$.

Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\beta)r - \sum_{k=2}^{\infty}\big[k-\beta+\alpha e^{i\theta}(k-1)\big][1+\lambda(k-1)]C(n,k)|a_k|r^k}{z - \sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n,k)|a_k|r^k}\right\} \geq 0.$$

Since $\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\beta)r - \sum_{k=2}^{\infty}\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)|a_k|r^k}{z - \sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n,k)|a_k|r^k}\right\} \geq 0.$$

Letting $r \to 1^-$, we get the desired result. Finally the result is sharp with the extremal function f given by

$$f(z) = z - \frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}z^n. \tag{5}$$

□

3. Growth and Distortion Theorems

Theorem 3.1. *Let the function f defined by (1) be in the class $M_\lambda^n(\alpha,\beta)$. Then for $|z| = r$ we have*

$$r - \frac{1-\beta}{(n+1)(2-\beta+\alpha)(1+\lambda)}r^2 \leq |f(z)| \leq r + \frac{1-\beta}{(n+1)(2-\beta+\alpha)(1+\lambda)}r^2. \tag{6}$$

Equality holds for the function

$$f(z) = z - \frac{1-\beta}{(n+1)(2-\beta+\alpha)(1+\lambda)}z^2. \tag{7}$$

Proof. We only prove the right hand side inequality in (6), since the other inequality can be justified using similar arguments. In view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{1-\beta}{(n+1)(2-\beta+\alpha)(1+\lambda)}.$$

Since, $f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k$

$$|f(z)| = |z| - \sum_{k=2}^{\infty}|a_k||z|^k \leq r + \sum_{k=2}^{\infty}|a_k|r^k$$

$$\leq r + r^2 \sum_{k=2}^{\infty} |a_k| \leq r + \frac{1-\beta}{(n+1)(2-\beta+\alpha)(1+\lambda)} r^2,$$

which yields the right hand side inequality of (6). □

Next, by using the same technique as in proof of Theorem 3.1, we give the distortion result.

Theorem 3.2. *Let the function f defined by (1) be in the class $M_\lambda^n(\alpha, \beta)$. Then for $|z| = r$ we have*

$$1 - \frac{2(1-\beta)}{(n+1)(2-\beta+\alpha)(1+\lambda)} r \leq |f'(z)| \leq 1 + \frac{2(1-\beta)}{(n+1)(2-\beta+\alpha)(1+\lambda)} r.$$

Equality holds for the function given by (7).

Theorem 3.3. *$f \in M_\lambda^n(\alpha, \beta)$, then $f \in T^*(\gamma)$, where*

$$\gamma = 1 - \frac{(k-1)(1-\beta)}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k) - (1-\beta)}.$$

The result is sharp, with function given by (7).

Proof. It is sufficient to show that (3) implies $\sum\limits_{k=2}^{\infty} (k-\gamma)|a_k| \leq 1-\gamma$, that is,

$$\frac{k-\gamma}{1-\gamma} \leq \frac{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{1-\beta}, \text{ then}$$

$$\gamma \leq 1 - \frac{(k-1)(1-\beta)}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k) - (1-\beta)}.$$

The above inequality holds true for $n \in \mathbb{N}_0$, $k \geq 2, \alpha, \lambda \geq 0$ and $0 \leq \beta < 1$. □

4. Extreme points

Theorem 4.1. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} z^k, \quad (k \geq 2).$$

Then $f \in M_\lambda^n(\alpha, \beta)$, if and only if it can be represented in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \quad \left(\mu_k \geq 0, \quad \sum_{k=1}^{\infty} \mu_k = 1\right). \tag{8}$$

Proof. Suppose $f(z)$ can be expressed as in (8). Then

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

$$= \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k \left\{ z - \frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} z^k \right\}$$

$$= \mu_1 z + \sum_{k=2}^{\infty} \mu_k z - \sum_{k=2}^{\infty} \mu_k \left\{ \frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} z^k \right\}$$

$$= z - \sum_{k=2}^{\infty} \mu_k \frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} z^k.$$

Thus

$$= \sum_{k=2}^{\infty} \mu_k \left(\frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} \right) \times \left(\frac{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{1-\beta} \right)$$

$$= \sum_{k=2}^{\infty} \mu_k = \sum_{k=1}^{\infty} \mu_k - \mu_1 = 1 - \mu_1 \leq 1.$$

So by Theorem 2.1, $f \in M_\lambda^n(\alpha, \beta)$.

Conversely, we suppose $f \in M_\lambda^n(\alpha, \beta)$. Since

$$|a_k| \leq \frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} \quad k \geq 2.$$

We may set

$$\mu_k = \frac{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{1-\beta} |a_k| \quad k \geq 2.$$

and $\mu_1 = 1 - \sum\limits_{k=2}^{\infty} \mu_k$. Then

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k = z - \sum_{k=2}^{\infty} \mu_k \frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} z^k$$

$$= z - \sum_{k=2}^{\infty} \mu_k [z - f_k(z)] = z - \sum_{k=2}^{\infty} \mu_k z + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

$$= \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) = \sum_{k=1}^{\infty} \mu_k f_k(z).$$

□

Corollary 4.2. *The extreme points of $M_\lambda^n(\alpha, \beta)$ are the functions*

$$f_1(z) = z \text{ and } f_k(z) = z - \frac{1-\beta}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} z^k, \quad k \geq 2.$$

5. Radii of Close-to-convexity, Starlikeness and Convexity

A function $f \in M_\lambda^n(\alpha, \beta)$ is said to be close-to-convex of order δ if it satisfies

$$\Re\{f'(z)\} > \delta, \quad (0 \le \delta < 1; z \in \mathbb{U}).$$

Also a function $f \in M_\lambda^n(\alpha, \beta)$ is said to be starlike of order δ if it satisfies

$$\Re\frac{zf'(z)}{f(z)} > \delta, \quad (0 \le \delta < 1; z \in \mathbb{U}).$$

Further a function $f \in M_\lambda^n(\alpha, \beta)$ is said to be convex of order δ if and only if $zf'(z)$ is starlike of order δ, that is if

$$\Re\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \delta, \quad (0 \le \delta < 1; z \in \mathbb{U}).$$

Theorem 5.1. *Let $f \in M_\lambda^n(\alpha, \beta)$. Then f is close-to-convex of order δ in $|z| < R_1$, where*

$$R_1 = \inf_{k\ge 2}\left[\frac{(1-\delta)\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{k(1-\beta)}\right]^{\frac{1}{k-1}}.$$

The result is sharp with the extremal function f given by (5).

Proof. It is sufficient to show that $|f'(z) - 1| \le 1 - \delta$ for $|z| < R_1$. We have

$$|f'(z) - 1| = \left| - \sum_{k=2}^{\infty} k a_k z^{k-1}\right| \le \sum_{k=1}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \le 1 - \delta$ if

$$\sum_{k=2}^{\infty}\left(\frac{k}{1-\delta}\right)|a_k||z|^{k-1} \le 1. \tag{9}$$

But Theorem 2.1 confirms that

$$\sum_{k=2}^{\infty}\frac{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{1-\beta}|a_k| \le 1. \tag{10}$$

Hence (9) will be true if $\frac{k|z|^{k-1}}{1-\delta} \le \frac{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{1-\beta}$.

We obtain

$$|z| \le \left\{\frac{(1-\delta)\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{k(1-\beta)}\right\}^{\frac{1}{k-1}}, \quad (k \ge 2)$$

as required. □

Theorem 5.2. *Let $f \in M_\lambda^n(\alpha,\beta)$. Then f is starlike of order δ in $|z| < R_2$, where*

$$R_2 = \inf_{k\geq 2}\left[\frac{(1-\delta)\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{(k-\delta)(1-\beta)}\right]^{\frac{1}{k-1}}.$$

The result is sharp with the extremal function f given by (5).

Proof. We must show that $\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\delta$ for $|z| < R_2$. We have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{-\sum\limits_{k=2}^{\infty}(k-1)a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k z^{k-1}}\right| \leq \frac{\sum\limits_{k=2}^{\infty}(k-1)|a_k||z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}|a_k||z|^{k-1}} \leq 1-\delta. \tag{11}$$

Hence (11) holds true if $\sum\limits_{k=2}^{\infty}(k-1)|a_k||z|^{k-1} \leq (1-\delta)\left\{1-\sum\limits_{k=2}^{\infty}|a_k||z|^{k-1}\right\}$ or, equivalently,

$$\sum_{k=2}^{\infty}\frac{(k-\delta)}{(1-\delta)}|a_k||z|^{k-1} \leq 1. \tag{12}$$

Hence, by using (10) and (12) will be true if

$$\frac{(k-\delta)}{(1-\delta)}|z|^{k-1} \leq \frac{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{1-\beta}$$

or if

$$|z| \leq \left\{\frac{(1-\delta)\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{(k-\delta)(1-\beta)}\right\}^{\frac{1}{k-1}}, \quad (k\geq 2)$$

which completes the proof. □

Theorem 5.3. *Let $f \in M_\lambda^n(\alpha,\beta)$. Then f is convex of order δ in $|z| < R_3$, where*

$$R_3 = \inf_{k\geq 2}\left[\frac{(1-\delta)\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}{k(k-\delta)(1-\beta)}\right]^{\frac{1}{k-1}}.$$

The result is sharp with the extremal function f given by (5).

Proof. By using the same technique in the proof of Theorem 5.2, we can show that $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\delta$ for $|z| \leq R_3$, with the aid of Theorem 2.1. Thus we have the assertion of Theorem 5.3. □

6. Inclusion theorem involving modified Hadamard products

For functions

$$f_j(z) = z - \sum_{k=2}^{\infty} |a_{k,j}| z^k \quad (j = 1, 2) \tag{13}$$

in the class $\mathcal{A}$, we define the modified Hadamard product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ given by $f_1(z) * f_2(z) = z - \sum_{k=2}^{\infty} |a_{k,1}||a_{k,2}| z^k$. We can prove the following.

Theorem 6.1. *Let the functions $f_j(z)$ $(j = 1, 2)$ given by (13) be on the class $M_\lambda^n(\alpha, \beta)$ respectively. Then $(f_1 * f_2)(z) \in M_\lambda^n(\alpha, \xi)$, where*

$$\xi = 1 - \frac{(1-\beta)^2}{(n+1)(2-\beta)(2-\beta+\alpha)(1+\lambda) - (1-\beta)^2}.$$

Proof. Employing the technique used earlier by Schild and Silverman [7], we need to find the largest ξ such that

$$\sum_{k=2}^{\infty} \frac{\big[k - \xi + \alpha(k-1)\big][1 + \lambda(k-1)]C(n,k)}{1-\xi} |a_{k,1}||a_{k,2}| \leq 1.$$

Since $f_j(z) \in M_\lambda^n(\alpha, \beta)$ $(j = 1, 2)$, then we have

$$\sum_{k=2}^{\infty} \frac{\big[k - \beta + \alpha(k-1)\big][1 + \lambda(k-1)]C(n,k)}{1-\beta} |a_{k,1}| \leq 1,$$

and

$$\sum_{k=2}^{\infty} \frac{\big[k - \beta + \alpha(k-1)\big][1 + \lambda(k-1)]C(n,k)}{1-\beta} |a_{k,2}| \leq 1,$$

by the Cauchy-Schwartz inequality, we have

$$\sum_{k=2}^{\infty} \frac{\big[k - \beta + \alpha(k-1)\big][1 + \lambda(k-1)]C(n,k)}{1-\beta} \sqrt{|a_{k,1}||a_{k,2}|} \leq 1.$$

Thus it is sufficient to show that

$$\begin{aligned} &\frac{\big[k - \xi + \alpha(k-1)\big][1 + \lambda(k-1)]C(n,k)}{1-\xi} |a_{k,1}||a_{k,2}| \\ &\leq \frac{\big[k - \beta + \alpha(k-1)\big][1 + \lambda(k-1)]C(n,k)}{1-\beta} \sqrt{|a_{k,1}||a_{k,2}|} \quad (k \geq 2), \end{aligned}$$

that is,

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{(1-\xi)\big[k - \beta + \alpha(k-1)\big]}{(1-\beta)\big[k - \xi + \alpha(k-1)\big]}.$$

Note that

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{(1-\beta)}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)}.$$

Consequently, we need only to prove that

$$\frac{(1-\beta)}{\big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)} \leq \frac{(1-\xi)\big[k-\beta+\alpha(k-1)\big]}{(1-\beta)\big[k-\xi+\alpha(k-1)\big]} \quad (k \geq 2),$$

or, equivalently, that

$$\xi \leq 1 - \frac{(k-1)(1+\alpha)(1-\beta)^2}{\big[k-\beta+\alpha(k-1)\big]^2[1+\lambda(k-1)]C(n,k)-(1-\beta)^2} \quad (k \geq 2).$$

Since

$$A(k) = 1 - \frac{(k-1)(1+\alpha)(1-\beta)^2}{\big[k-\beta+\alpha(k-1)\big]^2[1+\lambda(k-1)]C(n,k)-(1-\beta)^2} \quad (k \geq 2).$$

is an increasing function of $k (k \geq 2)$, letting $k = 2$ in last equation, we obtain

$$\xi \leq A(2) = 1 - \frac{(1+\alpha)(1-\beta)^2}{\big[2-\beta+\alpha\big]^2(1+\lambda)(n+1)-(1-\beta)^2}.$$

Finally, by taking the function given by (7). we can see that the result is sharp. □

7. Convolution and Integral Operators

Let $f(z)$ be defined by (1), and suppose that $g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k$. Then,the Hadamard product (or convolution) of $f(z)$ and $g(z)$ defined here by

$$f(z) * g(z) = (f * g)(z) = z - \sum_{k=2}^{\infty} |a_k||b_k| z^k.$$

Theorem 7.1. *Let* $f \in M_\lambda^n(\alpha,\beta)$*, and* $g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k \quad (0 \leq |b_n| \leq 1)$*. Then* $f * g \in M_\lambda^n(\alpha,\beta)$

Proof. In view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)|a_k||b_k|$$
$$\leq \sum_{k=2}^{\infty} \big[k-\beta+\alpha(k-1)\big][1+\lambda(k-1)]C(n,k)|a_k| \leq (1-\beta).$$

□

Theorem 7.2. *Let $f \in M_\lambda^n(\alpha, \beta)$ and let v be real number such that $v > -1$, then the function $F(z) = \frac{v+1}{z^v} \int\limits_0^z t^{v-1} f(t)dt$ also belongs to the class $M_\lambda^n(\alpha, \beta)$.*

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k, \text{ where } A_k = \left(\frac{v+1}{v+k}\right) |a_k|.$$

Since $v > -1$, than $0 \leq A_k \leq |a_k|$. Which in view of Theorem 2.1, $F \in M_\lambda^n(\alpha, \beta)$. $\square$

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School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600 Selangor D. Ehsan, Malaysia
E-mail address: ommath@hotmail.com